

Twisting $K3 \times T^2$ orbifolds

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ABSTRACT: We construct a class of geometric twists of Calabi-Yau manifolds of Voisin-Borcea type $(K3 \times T^2)/\mathbb{Z}_2$ and study the superpotential in a type IIA orientifold based on this geometry. The twists modify the direct product by fibering the $K3$ over T^2 while preserving the \mathbb{Z}_2 involution. As an important application, the Voisin-Borcea class contains $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, the usual setting for intersecting D6 brane model building. Past work in this context considered only those twists inherited from T^6 , but our work extends these twists to a subset of the blow-up modes. Our work naturally generalizes to arbitrary $K3$ fibered Calabi-Yau manifolds and to nongeometric constructions.

KEYWORDS: Superstring Vacua, Supersymmetric Effective Theories, Intersecting branes models, Flux compactifications.

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1. Introduction

In the quest for a better understanding of the space of string theory vacua, it has become increasingly clear that in addition to Neveu-Schwarz and Ramond-Ramond fluxes, the set of discrete data defining a string compactification also includes a set of geometric (and nongeometric) twists. The geometric twists characterize the departure of the internal

manifold from special holonomy, in most cases from a Calabi-Yau threefold. One way to describe this departure is in terms of G -structures and intrinsic torsion. For $SU(3)$ structure, this means specifying the moduli-dependent quantities dJ and $d\Omega$. However, this description masks the discreteness of the underlying geometric choice. An alternative description without this deficiency is to define the twists as a discrete deformation of the Calabi-Yau cohomology ring (cf. section 2). In this approach, a construction of the resulting non Calabi-Yau manifold which realizes the deformed cohomology has been lacking except in a few special cases — the mirror of the quintic hypersurface in \mathbb{P}^4 [41],¹ certain T^2 fibrations over $K3$,², and the untwisted sector of the $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbifold [9, 1, 42, 16]. Therefore, one would like a more general construction. In addition, given the ubiquity of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ in intersecting brane models of low energy particle physics, it is of particular interest to understand how blow-up modes of this orbifold can participate in the geometric twists even this special case.

In this article, we interpret $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ as $(K3 \times T^2)/\mathbb{Z}_2$ and show how the full \mathbb{Z}_2 projected sector of the latter can be included in the choice geometric twists. From the point of view of the $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbifold, this corresponds to the untwisted sector plus one third of the twisted sector, since 16 of the 48 blow-up modes of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ are accounted for by the blow-up modes of $K3 = T^4/\mathbb{Z}_2$. Our construction generalizes straightforwardly to other Calabi-Yau threefolds of the Voisin-Borcea type (i.e., $(K3 \times T^2)/\mathbb{Z}_2$ for other choices of \mathbb{Z}_2 involution) [44, 48], and more abstractly to arbitrary $K3$ fibered Calabi-Yau manifolds and nongeometric constructions.

An outline of the paper is as follows.

In section 2, we define geometric twists as discrete data that deform the closure and non-exactness properties of the generators of the Calabi-Yau integer cohomology ring.

Sections 3 and 4 are the core of the paper. In section 3 and its associated Apps. A and B, we consider the orbifold $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, its interpretation as $(K3 \times T^2)/\mathbb{Z}_2$, and the discrete twist data needed to parametrize a nontrivial $K3$ fibration over T^2 compatible with the \mathbb{Z}_2 involution. Along the way, we give a pedagogical review of the coset moduli space of $K3$ and its interpretation as a change of basis between two natural bases of $H^2(K3, \mathbb{R})$. We also discuss the quantization of the twist data and how it differs in two qualitatively different classes of $K3$ fibrations

In section 4, we consider an $\mathcal{N} = 1$ orientifold of type IIA string theory compactified on this twisted background. After describing the formal structure of the 4D $\mathcal{N} = 1$ supergravity theory — superpotential, Kähler potential, and Bianchi identities associated with D6 charge — we turn to an analysis of the vacua of the theory, focusing on supersymmetric Minkowski vacua. We find that supersymmetric Minkowski vacua are of the holomorphic monopole form discussed in refs. [34, 20]. These type IIA orientifold vacua formally lift to

¹To be fair, ref. [41] has given a formal description that applies not only to the the quintic but to any T^3 fibered Calabi-Yau 3-fold X . However, to apply this formalism to a given choice of X , the Strominger-Yau-Zaslow T^3 fibration needs to be explicitly known for X .

²Heterotic vacua involving T^2 fibrations over $K3$ were first discussed in ref. [14]. A more recent and thorough account can be found in ref. [3]. See also ref. [36] for type IIA compactifications based on this geometry.

compactifications of M theory on manifolds of G_2 holonomy, which could in principle be studied directly. However, in the past, the dual type IIA D6/O6 perspective has proven a fruitful setting for model building. Enriching this framework through the introduction of new geometric twist data provides what we hope will be a useful addition to the model building toolbox. After discussing the validity of the classical supergravity description, we present three examples of Minkowski vacua which share the property that all blow-up moduli of $(K3 \times T^2)/\mathbb{Z}_2$ are driven to zero. The first of the examples includes the full metric backreaction and dilaton profile in order to illustrate that these features do not affect the supersymmetry constraints on moduli. The section ends with a short discussion of the supersymmetry conditions in the more generic case of Anti de Sitter vacua.

Sections 5 and 6 are devoted to generalizations. In section 5 and its associated appendix C, we describe a straightforward generalization from $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ to all Calabi-Yau manifolds of the Voisin-Borcea type $(K3 \times T^2)/\mathbb{Z}_2$. In fact, it was primarily notational simplicity and the centrality of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ in model building that motivated the focus on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ in section 3 and 4. The other Voisin-Borcea manifolds would have served equally well. In section 6, we describe further generalizations, first to the class of $K3$ fibrations over \mathbb{P}^1 , and then to nongeometric twists that employ the full $\Gamma_{4,20}$ duality group of type IIA string theory on $K3$.

In section 7, we conclude with a brief discussion of our results and a description of possible avenues for future work.

2. Geometric twists as discrete deformation of Calabi-Yau cohomology

Let us start with an arbitrary Calabi-Yau manifold X . For simplicity, we assume that the torsion part of the cohomology of X vanishes, $H^{\text{tor}}(X, \mathbb{Z}) = 0$. (Once we introduce geometric twists this will change, and the discretely deformed manifold will generically no longer be a Calabi-Yau.) Let ω_a and $\tilde{\omega}^a$ denote bases of $H^2(X, \mathbb{Z})$ and $H^4(X, \mathbb{Z})$ respectively, where the two bases have been chosen to satisfy

$$\int \omega_a \wedge \tilde{\omega}^b = \delta_a^b. \tag{2.1}$$

Likewise, we choose a symplectic decomposition of $H_3(X, \mathbb{Z})$ into A -cycles and B -cycles, and corresponding decomposition of the cohomology $H^3(X, \mathbb{Z})$ into α_A and β^A , where

$$\int \alpha_A \wedge \beta^B = \delta_A^B. \tag{2.2}$$

For the Calabi-Yau manifold X , all of these basis forms are closed, by the definition of cohomology. To geometrically twist X , we introduce relations analogous to the defining closure relation for a twisted torus: $d\eta^m + \frac{1}{2}\gamma^m_{np}\eta^n \wedge \eta^p = 0$. Since we have no global 1-forms on a Calabi-Yau, the geometric twist must be expressed in terms of ω_a , $\tilde{\omega}^a$, α_A and β^A .

Suppose that we modify the Calabi-Yau cohomology by introducing the twisted closure relations

$$d\omega_a = -M_a^A \alpha_A + N_{aA} \beta^A, \tag{2.3}$$

while retaining the condition that $\alpha_A \wedge \omega_a$ and $\beta^A \wedge \omega_a$ be cohomologically trivial. Here, M_a^A and N_{aA} are $b_2 \times \frac{1}{2}b_3$ integer matrices (up to an overall scale factor, cf. section 3.3), where b_n denotes the n th Betti number. Then,

$$\begin{aligned} N_{aA} &= \int \alpha_A \wedge d\omega_a = - \int (d\alpha_A) \wedge \omega_a, \\ M_a^A &= \int \beta^A \wedge d\omega_a = - \int (d\beta^A) \wedge \omega_a. \end{aligned} \tag{2.4}$$

From the second equality in each line,

$$d\alpha_A = -N_{aA}\tilde{\omega}^a \quad \text{and} \quad d\beta^A = -M_a^A\tilde{\omega}^a, \tag{2.5}$$

respectively. So, the twisting of α_A and β^A is determined from that of ω_a without any additional data. The consistency condition $d^2\omega_a = 0$ is

$$NM^T - MN^T = 0. \tag{2.6}$$

As already mentioned in the introduction, while this prescription for discrete deformation of the Calabi-Yau cohomology is straightforward, it does not amount to a construction of a new non Calabi-Yau manifold \tilde{X} realizing the modified cohomology groups. It is not clear that such a manifold actually exists for arbitrary choice of discrete twist data M_a^A and N_{aA} satisfying eq. (2.6). A goal of this paper is to explicitly construct \tilde{X} for the Voisin-Borcea class of Calabi-Yau manifolds X (with special emphasis on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$) and for a subset of twists M_a^A and N_{aA} .

Note that we can express eqs. (2.3) through (2.6) in a manifestly symplectically covariant manner if we define a vector

$$\alpha = \begin{pmatrix} \alpha_A \\ \beta^A \end{pmatrix}, \tag{2.7}$$

and matrices

$$\begin{aligned} \mathbf{M} &= \begin{pmatrix} N_{aA} & M_a^A \end{pmatrix}, \\ \mathbf{\Lambda} &= \begin{pmatrix} 0 & -\delta_A^B \\ \delta_A^B & 0 \end{pmatrix}. \end{aligned} \tag{2.8}$$

Then eqs. 2.3 and 2.5 can be written

$$d\omega = \mathbf{M}\mathbf{\Lambda}\alpha \quad \text{and} \quad d\alpha^T = \tilde{\omega}^T\mathbf{M}.$$

The consistency condition $d^2\omega_a = 0$ of eq. (2.6) becomes $\mathbf{M}\mathbf{\Lambda}\mathbf{M}^T = 0$.

It is clear that the geometric twists lift a part of the cohomology ring of the original Calabi-Yau manifold X , since the forms on the r.h.s. of eqs. (2.3) and (2.5) are now exact, while those appearing on the l.h.s. now fail to be closed. The precise statement is that the free (non-torsion) part of the cohomology is reduced from

$$H_{\text{free}}^3(X) = \mathbb{Z}^{b^3(X)} \quad \text{and} \quad H_{\text{free}}^4(X) = \mathbb{Z}^{b^2(X)} \quad \text{for the Calabi-Yau,}$$

to

$$H_{\text{free}}^3(\tilde{X}) = \mathbb{Z}^{b^3(X) - \text{rank}(\mathbf{M})} \quad \text{and} \quad H_{\text{free}}^4(\tilde{X}) = \mathbb{Z}^{b^2(X) - \text{rank}(\mathbf{M})}$$

after introducing the geometric twists. Assuming that the original Calabi-Yau manifold X has trivial torsion, the torsion part of the cohomology of the twisted manifold \tilde{X} is³ $H_{\text{tor}}^3(\tilde{X}) = H_{\text{tor}}^4(\tilde{X}) = \bigotimes_{i=1}^{\text{rank}(\mathbf{M})} \mathbb{Z}_{m_i}$, where the m_i are determined by the particular choice of the matrix \mathbf{M} .

The interpretation of the Calabi-Yau cohomology $H^*(X, \mathbb{Z})$ from the point of view of the twisted manifold \tilde{X} is as a “first approximation” to the cohomology $H^*(\tilde{X}, \mathbb{Z})$ in a sense that has been made precise by Tomasiello [41] using spectral sequences. In this description, the cohomology of X corresponds to the term $E_2^{p,q}$ in a Leray-Hirsch spectral sequence. After a finite number of steps (which in this context we expect to be just one more) the sequence converges to the fixed value $E_\infty^{p,q}$, which gives the cohomology of \tilde{X} .

3. Geometric twists of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$

We now specialize to the Calabi-Yau orbifold $X = T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. There are two versions of this orbifold and we restrict to the choice with trivial discrete torsion. This choice gives topology $(h^{1,1}, h^{2,1}) = (51, 3)$. For the other choice, the Hodge numbers are reversed.

The two \mathbb{Z}_2 involutions each invert a T^4 within the T^6 :

$$\begin{aligned} \sigma_3: & \quad (x^1, x^2; x^3, x^4; x^5, x^6) \rightarrow (-x^1, -x^2; -x^3, -x^4; +x^5, +x^6), \\ \sigma_1: & \quad (x^1, x^2; x^3, x^4; x^5, x^6) \rightarrow (+x^1, +x^2; -x^3, -x^4; -x^5, -x^6). \end{aligned} \quad (3.1)$$

The composition $\sigma_2 = \sigma_3 \otimes \sigma_1$ inverts a third T^4 . Let $T_{(\alpha)}^2$ denote the torus covered by coordinates $(x^{2\alpha-1}, x^{2\alpha})$ for $\alpha = 1, 2, 3$; and let $T_{(\alpha)}^4$ be the corresponding 4-torus covered by the complementary set of coordinates. The involutions have been labeled so that σ_α leaves $T_{(\alpha)}^2$ invariant, but inverts $T_{(\alpha)}^4$.

3.1 Topology of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$

A part of the (co)homology of $X = T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is inherited directly from the T^6 . Let us focus on the divisors. The subgroup of $H_4(X, \mathbb{Z})$ inherited from the T^6 is generated by “sliding divisors” made up of \mathbb{Z}_2 invariant pairs of 4-tori on the T^6 :

$$2F_\alpha = T_{(\alpha)}^4 \times \{(a, b) \cup (-a, -b)\} \quad \text{for } (a, b) \in T_{(\alpha)}^2 \text{ not a } \mathbb{Z}_2 \text{ fixed point.} \quad (3.2)$$

(The factor of 2 anticipates that F_α is also an element of $H_4(X, \mathbb{Z})$ as we discuss below.) The Poincaré dual cohomology generators are $2\omega_\alpha$, where

$$\omega_1 = 2dx^1 \wedge dx^2, \quad \omega_2 = 2dx^3 \wedge dx^4, \quad \omega_3 = 2dx^5 \wedge dx^6. \quad (3.3)$$

In addition, there are 48 exceptional divisors $E_{\alpha I}$ with dual cohomology generators

$$\omega_{\alpha I}, \quad \text{where } \alpha = 1, 2, 3, \quad I = 1, \dots, 16. \quad (3.4)$$

³This result uses Poincaré duality and the Universal Coefficient Theorem [45] $H_{\text{tor}}^{n+1} = H_n^{\text{tor}}$.

These come from blowing up the 3×16 fixed lines $\mathbb{P}^1 = T_{(\alpha)}^2/\mathbb{Z}_2$ located at transverse coordinates each equal to 0 or $1/2$.

The classes $2F_\alpha$ and $E_{\alpha I}$ with integer coefficients, generate only a subgroup of $H_2(X, \mathbb{Z})$. This subgroup omits some of the divisors that arise in orbifold twisted sectors.⁴ We now describe these divisors as well.

First let us consider those divisors that arise in the orbifold twisted sector with respect to only a \mathbb{Z}_2 subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \sigma_1, \sigma_2, \sigma_3\}$. Performing the orbifold quotient in two steps gives three presentations of X of the form $(K3 \times T^2)/\mathbb{Z}_2$, one for each of the three choices $K3_{(\alpha)} = T_{(\alpha)}^4/\sigma_\alpha$. The α th presentation makes it clear that $K3_{(\alpha)}$ exists as a cycle in the Calabi-Yau manifold X as the generic fiber over base $\mathbb{P}^1 = T_{(\alpha)}^2/\mathbb{Z}_2$. The homology class of this divisor is half of the class of $T_{(\alpha)}^4$, i.e., F_α .

The classes F_α and $E_{\alpha I}$ with integer coefficients still generate only a subgroup of $H_2(X, \mathbb{Z})$. The remaining divisors are as follows. Before resolving the fixed points, $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ has 3×4 divisors of topology $\mathbb{P}^1 \times \mathbb{P}^1$ from the fixed planes $T_{(\alpha)}^4/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ located at each of the 4 fixed points in the transverse coordinates. These divisors persist after the resolution and represent homology classes of the form

$$\frac{1}{2}F_\alpha - \frac{1}{2}(\text{sum of eight } E_{\beta I}),$$

as is discussed in more detail in in appendix B. These divisors together with the F_α and $E_{\alpha I}$ generate all of $H_2(X, \mathbb{Z})$.

For $H^3(X, \mathbb{Z})$, the story is simpler. The only complex structure deformations of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ are those inherited from the three $T_{(\alpha)}^2$. The forms

$$2dx^i \wedge dx^j \wedge dx^k, \quad \text{where } i = 1, 2, \quad j = 3, 4, \quad k = 5, 6. \quad (3.5)$$

on T^6 give a basis for $H^3(X, \mathbb{Z})$. However, it should be noted that the “sliding 3-cycles” inherited from T^6 consist of $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant quadruples of 3-tori in T^6 . The dual cohomology classes are twice those that appear in eq. (3.5).

The presence of the 3×16 exceptional cycles $E_{\alpha I}$ and absence of exceptional complex structure deformations can be understood intuitively as follows. As discussed above, there are three ways to view the orbifold X as $(K3 \times T^2)/\mathbb{Z}_2$, corresponding to the three choices $K3_{(\alpha)} = T_{(\alpha)}^4/\sigma_\alpha$. Each $K3_{(\alpha)}$ has 16 exceptional cycles, corresponding to 16 hyperKähler deformations that smooth the singularities of $T_{(\alpha)}^4/\mathbb{Z}_2$. The $\omega_{\alpha I}$, for $I = 1, \dots, 16$, generate these hyperKähler deformations of $K3_{(\alpha)}$. Finally, each hyperKähler deformation of $K3$ is generated by three real moduli. Given a choice of complex structure on the $K3$, two of these moduli generate complex structure deformations and the remaining modulus generates a Kähler deformation. When we perform the second \mathbb{Z}_2 operation, the explicit \mathbb{Z}_2 of $(K3 \times T^2)/\mathbb{Z}_2$, we project out the exceptional complex structure deformations of the $K3_{(\alpha)}$ and retain the Kähler deformations. This leads to Hodge numbers $h^{1,1} = 3 + 3 \times 16 = 51$ and $h^{2,1} = 3$, in agreement with eqs. (3.3), (3.4) and (3.5).

⁴Unfortunately, the word “twisted” can refer to either a winding sector in an orbifold conformal field theory, or to a topology that has been discretely modified, i.e., in going from a product space to a fibration. Whenever we refer the former, we will use the words orbifold twisted sector.

3.2 Geometric twists in the orbifold untwisted sector

The only geometric twists of $X = T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ that have been discussed in the literature to date are those that are inherited from the twisted T^6 . The twisted T^6 is a parallelizable six-manifold with global 1-forms η^m , $m = 1, \dots, 6$ (sections of the frame bundle), satisfying $d\eta^m + \frac{1}{2}\gamma_{np}^m \eta^n \wedge \eta^p = 0$. As the name suggests, this can be viewed as a discretely deformed version of the ordinary T^6 , characterized by the twist data γ_{np}^m antisymmetric in lower indices.⁵ For the geometry to be well-defined, the γ_{np}^m must satisfy $\gamma_{mn}^m = 0$ (for the existence of a global volume form) together with the Jacobi identity (for $d^2 = 0$), as consequence of which they define the structure constants of a Lie algebra. If the corresponding Lie group is compact, then the twisted T^6 is defined by this Lie group. If it is noncompact, then subject to certain existence caveats, the twisted T^6 is defined as the coset of the Lie group by a discrete subgroup.

We now relate the γ_{np}^m to the matrices M and N of the previous section. As above, define $T_{(\alpha)}^2$ to be the torus covered by coordinates $(x^{2\alpha-1}, x^{2\alpha})$ for $\alpha = 1, 2, 3$. The components of γ_{np}^m that survive the orbifold projection are those with m, n, p each on a different $T_{(\alpha)}^2$, and those with all three indices on the same $T_{(\alpha)}^2$. For simplicity, we set the latter components to zero; then the constraint $\gamma_{mn}^m = 0$ is trivially satisfied. Finally, we define twisted analogs of the differential forms (3.3) and (3.5). These are globally defined differential forms on $\tilde{X} = (\text{twisted-}T^6)/\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$\begin{aligned}
 \omega_1 &= 2\eta^1 \wedge \eta^2, \\
 \omega_2 &= 2\eta^3 \wedge \eta^4, \\
 \omega_3 &= 2\eta^5 \wedge \eta^6, \\
 \tilde{\omega}^1 &= 2\eta^3 \wedge \eta^4 \wedge \eta^5 \wedge \eta^6, \\
 \tilde{\omega}^2 &= 2\eta^1 \wedge \eta^2 \wedge \eta^5 \wedge \eta^6, \\
 \tilde{\omega}^3 &= 2\eta^1 \wedge \eta^2 \wedge \eta^3 \wedge \eta^4,
 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
 \alpha_0 &= 2\eta^1 \wedge \eta^3 \wedge \eta^5, \\
 \alpha_1 &= -2\eta^1 \wedge \eta^4 \wedge \eta^6, \\
 \alpha_2 &= -2\eta^2 \wedge \eta^3 \wedge \eta^6, \\
 \alpha_3 &= -2\eta^2 \wedge \eta^4 \wedge \eta^5, \\
 \beta^0 &= -2\eta^2 \wedge \eta^4 \wedge \eta^6, \\
 \beta^1 &= 2\eta^2 \wedge \eta^3 \wedge \eta^5, \\
 \beta^2 &= 2\eta^1 \wedge \eta^4 \wedge \eta^5, \\
 \beta^3 &= 2\eta^1 \wedge \eta^3 \wedge \eta^6.
 \end{aligned} \tag{3.7}$$

⁵For the ordinary T^6 , we have $\gamma_{np}^m = 0$ and sections of the frame bundle are constant linear combinations of the closed coordinate 1-forms dx^m .

In terms of γ_{jk}^i , the matrices M and N are

$$M_a^A = \begin{pmatrix} \gamma_{53}^2 & -\gamma_{64}^2 & \gamma_{63}^1 & \gamma_{54}^1 \\ -\gamma_{51}^4 & -\gamma_{61}^3 & \gamma_{62}^4 & -\gamma_{52}^3 \\ \gamma_{31}^6 & \gamma_{41}^5 & \gamma_{32}^5 & -\gamma_{42}^6 \end{pmatrix}, \quad N_{aA} = \begin{pmatrix} -\gamma_{64}^1 & \gamma_{53}^1 & -\gamma_{54}^2 & -\gamma_{63}^2 \\ \gamma_{62}^3 & \gamma_{52}^4 & -\gamma_{51}^3 & \gamma_{61}^4 \\ -\gamma_{42}^5 & -\gamma_{32}^6 & -\gamma_{41}^6 & \gamma_{31}^5 \end{pmatrix}. \quad (3.8)$$

The consistency condition (2.6) can be satisfied by choosing, for example, $M_\alpha^A = 0$. In the case of intersecting D-brane models, this is a requirement rather than a choice, as ω_a , $\tilde{\omega}^a$ and β^A are odd while α_A is even under the orientifold \mathbb{Z}_2 operation $\Omega(-1)^{FL}\mathcal{I}_3$. Here, Ω is worldsheet parity, $(-1)^{FL}$ is left-moving fermion parity and

$$\mathcal{I}_3: (x^1, x^2; x^3, x^4; x^5, x^6) \rightarrow (x^1, -x^2; x^3, -x^4; x^5, -x^6). \quad (3.9)$$

3.3 Geometric twists in the orbifold twisted sector

In this section, we show how to introduce geometric twists involving 16 of the 48 blow-up modes of $X = T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ through the following trick: The 16 exceptional Kähler deformations of $K3 = T^4/\mathbb{Z}_2$ are in the untwisted sector of $(K3 \times T^2)/\mathbb{Z}_2$, but in the twisted sector of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Therefore, if we twist $K3 \times T^2$ by fibering the $K3$ surface over the T^2 while at the same time preserving the existence of a \mathbb{Z}_2 involution, then the \mathbb{Z}_2 quotient of the result is a twisted version of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. For generic fibration, the twists involve all 16 of the $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ blow-up modes inherited from the $K3$.

Moduli space of K3. To describe a smooth fibration of $K3$ over T^2 we simply allow the moduli of the $K3$ surface to vary over T^2 . The moduli space of metrics on $K3$ is the coset

$$\mathcal{M} = \mathbb{R}_{>0} \times (\text{SO}(3) \times \text{SO}(19)) \backslash \text{SO}(3, 19) / \Gamma_{3,19}. \quad (3.10)$$

Here, the first factor in eq. (3.10) is the overall volume of the $K3$ surface. The second factor is the choice of hyperKähler structure. Its explicit form arises as follows. The coset describes the space of positive signature 3-planes in $\mathbb{R}^{3,19}$. This space appears since the cohomology group $H^2(K3, \mathbb{R})$ has signature (3, 19) with respect to the inner product $(\omega_1, \omega_2) = \int \omega_1 \wedge \omega_2$. That is, there are three selfdual 2-forms and nineteen anti-selfdual 2-forms on $K3$. The choice of hyperKähler structure on $K3$ is the choice of positive 3-plane spanned by $J, \text{Re} \Omega_{(2)}, \text{Im} \Omega_{(2)}$ in $H^2(K3, \mathbb{R})$. Finally, the quotient by the discrete group $\Gamma_{3,19}$ accounts for the fact that automorphisms of the lattice $H^2(K3, \mathbb{Z})$ relate equivalent $K3$ surfaces.

The $(\text{SO}(3) \times \text{SO}(19)) \backslash \text{SO}(3, 19) / \Gamma_{3,19}$ coset can be parametrized as [37]

$$M^{ab} = \begin{pmatrix} G & -GC & -GA^T \\ -C^T G G^{-1} + a + C^T G C & A^T + C^T G A^T & \\ -AG & A + AGC & 1 + AGA^T \end{pmatrix}, \quad (3.11)$$

where $a^{ij} = A^{Li} A^{Lj}$ and $C^{ij} = B^{ij} + \frac{1}{2} A^{Li} A^{Lj}$. Here, G_{ij} is a metric on $T^3(x^1, x^2, x^3)$, with coordinate periodicity $x^i \simeq x^i + 1$; B^{ij} is an antisymmetric bivector T^3 with periodicity $B^{ij} \simeq B^{ij} + 1$; and $\frac{1}{2} A^{Li}$, $I = 1 \dots 16$, are the coordinates of sixteen points on the T^3 . The

index upper a runs over lower i , upper i and I . In addition to the periodicities listed, and of course the $SL(3, \mathbb{Z})$ equivalence of G_{ij} under change of T^3 lattice basis, there are further identifications of G, B, A under more general elements of $\Gamma_{3,19}$.

This coset description is well known and roughly speaking makes manifest the duality between M theory on $K3$ and the heterotic or type I string on T^3 . To be more precise, given our definitions of G, B, A (note the index placement, in particular), the natural duality relates M theory on $K3$ not to type I, but rather to type I'—the T-dual of type I on T^3 under inversion of all three directions of the T^3 . In this version of the duality, the $K3$ moduli $\frac{1}{2}A^{Ii}$ are identified with the transverse scalars to the sixteen D6-branes, and the $K3$ moduli B^{ij} are identified with $-\frac{1}{2}\epsilon^{ijk}C_{(1)k}$, where $C_{(1)}$ is the RR 1-form. We have chosen to parametrize the $K3$ moduli space in terms of scalars adapted to this duality frame rather than the heterotic/type I duality frame for the following reason: The torus twists γ_{jk}^i discussed in section 3.1 correspond to twists of the physical T^3 in type I' variables, but to twists of the dual T^3 in the heterotic/type I variables [50].

Among the possible presentations of $SO(3, 19)$, the coset representative M^{ab} leaves invariant the $SO(3, 3 + 16)$ metric in off-diagonal form,

$$\eta_{ab} = \begin{pmatrix} 0 & \delta^i_j & 0 \\ \delta_i^j & 0 & 0 \\ 0 & 0 & \delta_{IJ} \end{pmatrix}. \tag{3.12}$$

It can be expressed concisely as $M = V^T V$, where V_Λ^a is the vielbein

$$V(E, B, A) = \begin{pmatrix} E & -EC & -EA^T \\ 0 & E^{-1T} & 0 \\ 0 & A & 1 \end{pmatrix}, \tag{3.13}$$

and E^λ_i is the vielbein for the metric G_{ij} on T^3 . Here, Λ is the $SO(3) \times SO(19)$ coset index and a is the $SO(3, 19)$ index. If we choose to write A, B, C with vielbein indices instead of coordinate indices, then the last expression instead takes the form

$$\tilde{V}(E, B, A) = \begin{pmatrix} E & -CE^{-1T} & -A^T \\ 0 & E^{-1T} & 0 \\ 0 & AE^{-1T} & 1 \end{pmatrix}. \tag{3.14}$$

Cohomology of $K3$. The second cohomology group of $K3 = T^4/\mathbb{Z}_2$ is generated by the subgroup inherited from T^4 ,

$$\chi^i = 2dx^4 \wedge dx^i \quad \text{and} \quad \chi_i = \epsilon_{ijk} dx^j \wedge dx^k, \quad i, j, k = 1, 2, 3, \tag{3.15}$$

together with the blow-up modes

$$\chi_I, \quad I = 1, \dots, 16, \tag{3.16}$$

dual to the exceptional curves that arise from blowing up the sixteen \mathbb{Z}_2 fixed points.⁶ The cohomology classes $\chi_a = (\chi^i, \chi_i, \chi_I)$ satisfy

$$\int_{K3} \chi_a \wedge \chi_b = -2\eta_{ab}, \tag{3.17}$$

where η_{ab} is the $\text{SO}(3, 3 + 16)$ invariant metric introduced in eq. (3.12).⁷

Much like in the $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ discussion in section 3.1, the classes (3.15) and (3.16), with integer coefficients, do not generate all of $H^2(K3, \mathbb{Z})$, but rather an order two sublattice called the Kummer lattice. To obtain the full $H^2(K3, \mathbb{Z})$ we also need to include generators of the form

$$\frac{1}{2}f - \frac{1}{2}(\text{sum of four } \chi_I), \quad \text{where } f = \chi^i \text{ or } \chi_i,$$

dual to certain \mathbb{P}^1 divisors of $K3$. This is discussed in more detail in appendix B.

Cohomological interpretation of the coset matrix. The inverse M_{ab} of the coset representative eq. (3.11) has a simple interpretation in terms of the natural metric-dependent inner product on the χ_a :

$$\int_{K3} \star \chi_a \wedge \chi_b = 2M_{ab}. \tag{3.18}$$

If we define $\chi_\Lambda = V_\Lambda^a \chi_a$, then from the definition of the vielbein V_Λ^a , we have

$$\int_{K3} \star \chi_\Lambda \wedge \chi_{\Lambda'} = 2\delta_{\Lambda\Lambda'}. \tag{3.19}$$

So, the vielbein V_Λ^a takes the moduli-independent cohomology basis χ_a with moduli-dependent norm into a moduli-dependent basis χ_Λ with moduli-independent norm.⁸

Periodicities of $K3$ moduli. Define the $\text{SO}(3, 19)$ elements

$$V_1(x) = \begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1T} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V_2(y) = \begin{pmatrix} 1 & -y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V_3(z) = \begin{pmatrix} 1 & -\frac{1}{2}z^T z & -z^T \\ 0 & 1 & 0 \\ 0 & z & 1 \end{pmatrix}. \tag{3.20}$$

⁶In our basis, dx^4 plays a privileged role since we work in the convention that e^4 is the real part of the hyperKähler 1-form on T^4 . The forms $e^{1,2,3}$ are the three (hyper)imaginary parts. See appendix A for definitions of the e^i and a review of hyperKähler structure on T^4 .

⁷The change of basis from χ_a with intersection matrix $-2\eta_{ab}$ to χ'_a with the intersection matrix $-\text{Cartan}(E_8 \times E_8) \times U_{1,1}^3$ (referred to later in the Voisin-Borcea context) can be found in appendix B of ref. [25]. Unlike the χ_a , which require some half-integer coefficients to generate all of $H^2(K3, \mathbb{Z})$, the χ'_a generate $H^2(K3, \mathbb{Z})$ with purely integer coefficients. We thank K. Wendland for providing this reference.

⁸The attentive reader may have noticed that setting $A_n^{Ii} = 0$ in the coset matrix (3.13) should correspond to the $\text{SO}(32)$ point in moduli space. However, given our definitions, we obtain the $(A_1)^{16}$ point of the unresolved T^4/\mathbb{Z}_2 instead. The latter should correspond to evenly distributing the 16 A^{Ii} among the 8 fixed values where each of the $i = 1, 2, 3$ components is either 0 or 1/2. This problem is solved by first performing a change of basis $\chi \rightarrow \hat{\chi} = V(1, 1, -\Delta A)\chi$, in order to convert the $(A_1)^{16}$ adapted basis to the $\text{SO}(32)$ adapted basis, and then writing $\chi_\Lambda = V_\Lambda^a \hat{\chi}_a$. For notational simplicity, we leave this change of basis implicit.

Then, the vielbein $V(E, B, A)$ and $\tilde{V}(E, B, A)$ defined above are

$$V = V_1(E)V_2(B)V_3(A), \quad \tilde{V} = V_3(A)V_2(B)V_1(E). \quad (3.21)$$

From the interpretation $\chi_\Lambda = V_\Lambda^a \chi_a$ given in the last subsection, we obtain identifications of the $K3$ moduli from the automorphisms $\Gamma_{3,19}$ of $H^2(K3, \mathbb{Z})$. The group $\Gamma_{3,19}$ is a discrete subgroup of $\text{SO}(3, 19)$, which in turn contains, as a proper subgroup, the group $\text{SO}(3, 19, \mathbb{Z})$ of integer matrices Γ_a^b preserving the intersection matrix (3.17). (Since the χ_a require some half integer coefficients to generate all of $H^2(K3, \mathbb{Z})$, there are also elements of $\Gamma_{3,19}$ which contain half integer components, and are therefore not contained in $\text{SO}(3, 19; \mathbb{Z})$.) Let us focus on this subgroup. For any such automorphism $\Gamma_a^b \in \text{SO}(3, 19; \mathbb{Z})$ we obtain an identification $V(E, B, A)_\Lambda^b \simeq V(E, B, A)_\Lambda^a \Gamma_a^b$. From $V \simeq VV_3$ with $V_3(z) \in \text{SO}(3, 19; \mathbb{Z})$, we obtain the identifications $A^{iI} \simeq A^{iI} + 2$ for individual components A^{Ii} . Beyond this, for n -tuples of components there exist additional identifications, for example $(A^{Ii}, A^{Ji}) \simeq (A^{Ii} + 1, A^{Ji} + 1)$. From $V \simeq VV_2$ with $V_2(y) \in \text{SO}(3, 19; \mathbb{Z})$ we obtain the identifications $B^{ij} \simeq B^{ij} + 1$.

Fibration of $K3$ over T^2 . In section 3.2, we defined a twisted torus to be a parallelizable manifold, analogous to a torus, but with global 1-forms η^m satisfying $d\eta^m + \frac{1}{2}\gamma_{np}^m \eta^n \wedge \eta^p = 0$, which can be viewed as generalizations of the coordinate 1-forms dx^m on a torus. A special case of a twisted torus is a torus fibration over torus base. In this case, let us refine the index notation that we have been using, so that i, j, \dots denote fiber indices and m, n, \dots denote base indices. Then, for the fibration, we take $\eta^n = dx^n$ on the base, and have

$$d\eta^i + \gamma_{nj}^i dx^n \wedge \eta^j = 0, \quad (3.22)$$

for the 1-forms on the fiber.

Analogously, for a $K3$ fibration over $T^2(x^5, x^6)$, the global 2-forms on the fibration include twisted versions of the χ_a , which we will denote by ζ_a . The ζ_a satisfy

$$d\zeta_a + M_{na}^b dx^n \wedge \zeta_b = 0. \quad (3.23)$$

From the topological consistency condition $d^2 = 0$, we require $[M_m, M_n] = 0$. Let us define $\Gamma_5 = \exp(-x^5 M_5)$ and $\Gamma_6 = \exp(-x^6 M_6)$. The closure condition (3.23) follows by promoting the forms χ_a on $K3$ to global forms on the fibration through the relation

$$\zeta_a = (\Gamma_5(x^5)\Gamma_6(x^6))_a^b \chi_b. \quad (3.24)$$

In order to preserve the inner product (3.17) and integrality of the basis, the monodromy matrices $\Gamma_n(1)$ should be elements of $\Gamma_{3,19}$. The monodromy matrices, or equivalently, the Lie algebra elements M_5 and M_6 , completely determine the topology of the fibration.

Let us write

$$\Gamma_5(x^5) = \tilde{V}(\eta_{(5)}, \beta_5 x^5, m_5 x^5) \quad \text{and} \quad \Gamma_6(x^6) = \tilde{V}(\eta_{(6)}, \beta_6 x^6, m_6 x^6), \quad (3.25)$$

where vielbein \tilde{V} was defined in eq. (3.14). This ansatz describes the subset of $K3$ fibrations for which the $K3$ moduli B^{ij}, A^{Ii} undergo periodic linear shifts as we traverse the x^5 and

x^6 circles on the T^2 base. (Here, the reader may wish to refer to the discussion above on the cohomological interpretation of the coset vielbein). Here, β_n^{ij} and m_n^{Ii} are constant matrices, and the $\eta_{(n)j}^i(x)$ are given in terms of constants γ_{5j}^i and γ_{6j}^i as

$$\eta_{(5)j}^i = \exp(-\gamma_{5j}^i x^5) \quad \text{and} \quad \eta_{(6)j}^i = \exp(-\gamma_{6j}^i x^6). \quad (3.26)$$

Let us also write $\eta^i = \eta_{(5)j}^i \eta_{(6)k}^j dx^k$. Then, the η^i satisfy eq. (3.22), provided that γ_5 and γ_6 commute. And indeed, this is the case: $[\gamma_5, \gamma_6] = 0$, as a consequence of $[M_5, M_6] = 0$ (cf. the first condition in eq. (3.27) below).

In our discussion of the moduli space of $K3$, the coset matrix M^{ab} was defined in terms of tensors on a formal auxiliary T^3 . We can think of the η^i that we have just defined as a frame for a fibration of this formal T^3 over the physical T^2 that forms the base of the $K3$ fibration. The η^i encode the dependence of the $K3$ moduli G_{ij} on the base coordinates x^5, x^6 , as quantified by the data γ_{nj}^i . Likewise, β_n^{ij} and m_n^{Ii} parametrize the base coordinate dependence of the remaining $K3$ moduli B^{ij} and A^{Ii} , respectively.

Eqs. (3.24) and (3.25) together give the change of basis from local untwisted forms χ_b to the global twisted forms ζ_a on the $K3$ fibration. The reason that we have used the presentation \tilde{V} rather than V in (3.25) is that it simplifies the results below if we define β_n^{ij} and m_n^{Ii} to be tensor components with respect to the frame η^i rather than the coordinate 1-forms dx^i .

By differentiating eq. (3.25), we obtain

$$M_n = \begin{pmatrix} \gamma_n & \beta_n & m_n^T \\ 0 & -\gamma_n^T & 0 \\ 0 & -m_n & 0 \end{pmatrix}, \quad (3.27)$$

which determines the closure condition (3.23). In components, eq. (3.23) reads

$$\begin{aligned} d\zeta^i &= -dx^n \wedge (\gamma_{nj}^i \zeta^j + \beta_n^{ij} \zeta_j + m_n^{Ii} \zeta_I), \\ d\zeta_i &= dx^n \wedge \gamma_{ni}^j \zeta_j, \\ d\zeta_I &= dx^n \wedge \delta_{IJ} m_n^{Ji} \zeta_i, \end{aligned} \quad (3.28)$$

and the condition $[M_n, M_p] = 0$ becomes

$$\gamma_{[n} \gamma_{p]} = 0, \quad -\gamma_{[n} \beta_{p]} + \beta_{[n} \gamma_{p]}^T + m_{[n}^T m_{p]} = 0, \quad \gamma_{[n} m_{p]}^T = 0. \quad (3.29)$$

Quantization and elliptic versus parabolic twists. To define a $K3$ fibration over T^2 , the twist data γ_{nj}^i , β_n^{ij} and m_n^{Ii} must be chosen so that the monodromy matrices $\Gamma_5 = \exp(-x^5 M_5)$ and $\Gamma_6 = \exp(-x^6 M_6)$ are elements of $\Gamma_{3,19}$. In deriving the result (3.27), we have further restricted to a particularly simple subgroup of $\text{SO}(3, 19; \mathbb{Z})$ corresponding to the subgroup of $\text{SO}(3, 19)$ spanned by our choice of vielbein for the $(\text{SO}(3) \times \text{SO}(19)) \backslash \text{SO}(3, 19)$ coset representatives. For this subgroup, we now relate the condition $\Gamma_5, \Gamma_6 \in \text{SO}(3, 19; \mathbb{Z})$ to restrictions on the allowed twist data.

For the β_n^{ij} and m_n^{Ii} , there is no subtlety. The conditions $\beta_n^{ij} \in \mathbb{Z}$ and $m_n^{Ii} \in 2\mathbb{Z}$ lead to integer components in Γ_5, Γ_6 , and follow directly from the periodicities $B^{ij} \simeq B^{ij} + 1$

and $A^{iI} \simeq A^{iI} + 2$. For the γ_{nj}^i , consider the diagonal blocks $\eta_{(n)}$ and $\eta_{(n)}^{-1T}$ of Γ_n (cf. eq. (3.26)). The constraint that these be integer matrices gives rise to qualitatively different conditions, depending on the nilpotency properties of the γ_n (or equivalently, the idempotency properties $\eta_{(n)}$). We will consider only two special cases.

First, consider the case that $(\gamma_{(n)})^2 = 0$. We will refer to this as the “parabolic case” by analogy to the conjugacy classes of $SL(2, \mathbb{R})$.⁹ Then $\eta_{(5)j}^i = 1 - \gamma_{5j}^i x^5$ and $\eta_{(6)j}^i = 1 - \gamma_{6j}^i x^6$. In order that these be integer matrices at $x^n = 1$, we require $\gamma_{nj}^i \in \mathbb{Z}$.

In contrast, consider the case that $\gamma_{(n)}$ is not nilpotent of any degree, but instead $\eta_{(n)}$ satisfies $(\eta_{(n)})^k = 1$ for some finite positive integer k . We will refer to this as the “elliptic case,” again by analogy to $SL(2, \mathbb{R})$. For example, suppose that $\gamma_{(n)} = (\pi\lambda_{(n)}/2)\mathcal{I}$, for some integer matrix \mathcal{I} satisfying $\mathcal{I}^2 = -1$. Here $\lambda_{(n)}$ is a proportionality constant. Then,

$$\eta_{(5)}(x^5) = \exp(-\gamma_5 x^5) = \cos\left(\frac{\pi}{2}\lambda_{(5)}x^5\right) - \mathcal{I} \sin\left(\frac{\pi}{2}\lambda_{(5)}x^5\right), \quad (3.30)$$

with a similar expression for $\eta_{(6)}$. In this case, it is clear that for $\eta_{(n)}$ to be integer valued at $x^n = 1$, we require $\lambda_{(n)} \in \mathbb{Z}$, and hence $\gamma_{nj}^i \in \frac{\pi}{2}\mathbb{Z}$. Note that $\eta_{(n)}$ is then idempotent of degree four, $(\eta_{(n)})^4 = 1$.

The qualitative difference between the two cases is that the first requires energy, i.e., curvature, since some of the $K3$ moduli vary linearly as a function of the base coordinates. This case therefore gives rise to a fibration of $K3$ over T^2 that metrically is locally distinguishable from a direct product. The second case typically requires some of the $K3$ moduli to be fixed to \mathbb{Z}_k symmetric values. This does not require energy from spatially varying moduli, or curvature. So, in this case, there is a global distinction, but no local metric distinction between the fibration and $K3 \times T^2$. Unless we state otherwise, we will have the former, parabolic case in mind in the remainder of the paper.

Fibration of $K3$ over T^2 , with \mathbb{Z}_2 involution. Consider the \mathbb{Z}_2 involution σ_1 acting on $K3_{(3)} \times T^2_{(3)}$, and let us focus on this $K3$. The differential forms χ_3 and χ^3 are even under the action of σ_1 , while $\chi_{1,2}$ and $\chi^{1,2}$ are odd. Since we defined σ_1 in the context of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, the $K3$ blow-up modes χ_I are all even under σ_1 and all survive as Kähler moduli in the quotient Calabi-Yau. (For Voisin-Borcea Calabi-Yau manifolds based on other $K3$ involutions, the χ_I in general split into even and odd subsets, as explained in section 5.)

Given a cohomology element $\chi_a \in H^2(K3, \mathbb{Z})$, the hyperKähler deformations of $K3$ that this element generates is

$$\delta g_{mn} = \epsilon^{a(i)} J_{(i)m}{}^p(\chi_a)_{pn}.$$

Here, $\epsilon^{a(i)}$ is the deformation parameter and $J_{(i)}$ is the triple of almost complex structures. For example, for χ_I , we have $\epsilon^{I(i)} = \delta A^{Ii}$. Under the \mathbb{Z}_2 involution σ_1 , the almost complex

⁹The hyperbolic, elliptic and parabolic conjugacy classes of $SL(2, \mathbb{R})$ can be represented by matrices $\Gamma = \begin{pmatrix} \exp(-x) & 0 \\ 0 & \exp(x) \end{pmatrix}$, $\begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix}$ and $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, which are the exponentials $\exp(-\gamma x)$ for $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, respectively. See ref. [12] for a discussion of Scherk-Schwarz compactifications with $SL(2, \mathbb{Z})$ monodromy over S^1 .

structures $J_{(1)}$ and $J_{(2)}$ on the $K3$ are odd, while $J_{(3)}$ is even (cf. appendix A). So, with the choice $J = J_{(3)}$, the \mathbb{Z}_2 parities listed above tell us that the \mathbb{Z}_2 projection retains the moduli corresponding to the following \mathbb{Z}_2 -even metric deformations:

$$\begin{aligned} &\text{Kähler deformations generated by } \chi_3, \chi^3 \text{ and } \chi_I, \\ &\text{Complex structure deformations generated by } \chi_{1,2} \text{ and } \chi^{1,2}. \end{aligned} \tag{3.31}$$

Now, let us include the T^2 factor. The geometric twists are implemented by replacing the product $K3 \times T^2$ with a nontrivial fibration of $K3$ over T^2 . The topology of an arbitrary smooth oriented fibration of $K3$ over T^2 is characterized by the two group elements $\Gamma_5(1), \Gamma_6(1) \in \Gamma_{3,19}$ under which the $K3$ moduli are identified as one takes $x^5 \rightarrow x^5 + 1$, $x^6 \rightarrow x^6 + 1$, respectively. For the parabolic subset of these fibrations discussed in the previous section, the coset parameters E, B, A of eqs. (3.11) and (3.13) simply shift through an integer number of periods as we traverse the x^5 or x^6 circles.¹⁰ Let us focus on this subset of fibrations, for which the coset parametrization (3.11) is particularly adapted. Then, for example, for the blow-up modes, we write

$$A^{Ii} = A_0^{Ii} + m_n^{Ii} x^n, \quad n = 5, 6, \quad m_n^{Ii} \in 2\mathbb{Z}. \tag{3.32}$$

The first component A_0^{Ii} is the modulus component, the 0-mode on T^2 with continuous deformations δA^{Ii} . The second component $m_n^{Ii} x^n$ depends explicitly on the T^2 coordinates and is the discrete twist; it implements the even integer modular shifts $\Gamma_n(1)$: $A^{Ii} \rightarrow A^{Ii} + m_n^{Ii}$ for $x^n \rightarrow x^n + 1$.

This type of generalized compactification, in which the moduli from the first stage of the compactification (in this case, the $K3$) are given dependence on the remaining compactification coordinates (in this case, the T^2) in order to implement nontrivial twists by the action of the modular group, is referred to as a Scherk-Schwarz compactification [38, 33]. Generic Scherk-Schwarz compactifications involves all of the moduli from first stage of the compactification and not just the subset coming from the metric. Hence, they are more general than geometric fibrations of this section. In section 6.2, we will briefly mention nongeometric Scherk-Schwarz compactifications based on the full $\Gamma_{4,10}$ modular group of type IIA on $K3$ instead of the $\Gamma_{3,19}$ metric modular group considered here.

Since we wish to implement geometric twists of the Calabi-Yau $X = (K3 \times T^2)/\mathbb{Z}_2$, and not just of $K3 \times T^2$, the moduli and twists just discussed must be restricted to those that respect the \mathbb{Z}_2 isometry. The moduli survive for the even $K3$ metric deformations, as already described in (3.31). In addition, the discrete Scherk-Schwarz twists survive for the complementary set of odd metric deformations, since these depend linearly on the odd T^2 coordinates x^n . Thus, from the blow-up modes of $K3$, we obtain the following moduli and discrete geometric twists of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2) = (K3 \times T^2)/\mathbb{Z}_2$:

$$\begin{aligned} \text{Moduli:} & \quad \delta A^{I3}, \\ \text{Twists:} & \quad A^{Ii} = m_n^{Ii} x^n, \quad i = 1, 2, \quad n = 5, 6, \quad m_n^{Ii} \in 4\mathbb{Z}. \end{aligned} \tag{3.33}$$

¹⁰In the case of the vielbein E , this means that the quantities a^i_j that appear in the T^3 basis forms $E^i \propto dx^i + a^i_j dx^j$ for $i = 1, 2, 3$ (cf. the last subsection and appendix A) shift by integers.

As should be clear by now, the integers m_n^{Ii} partially characterize the fibration of $K3$ over T^2 . eq. (3.33) indicates which components are compatible with the \mathbb{Z}_2 isometry. In addition, the subset of γ_n^i and β_n^{ij} that respect the \mathbb{Z}_2 isometry are those for which one of i, j is equal to 3 and the other is equal to 1 or 2. Comparing to section 3.2, the m_n^{Ii} are genuinely new discrete twists of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ over those inherited from the twisted T^6 . However, the β_n^{ij} are just the T^6 twists in disguise. This is most apparent in the orbifold limit $K3 = T^4/\mathbb{Z}_2$. As discussed in appendix A, the interpretation of the modulus B^{ij} is as the flat connection $a^4_i = -\frac{1}{2}\epsilon_{ijk}B^{jk}$ which appears in 1-form $e^4 = R_4(dx^4 + a^4_j dx^j)$ needed to complete the $T^3(x^1, x^2, x^3)$ into a T^4 . Thus $\gamma_{ni}^4 = \frac{1}{2}\epsilon_{ijk}\beta_n^{jk}$. Finally, the integers γ_{nj}^i , β_n^{ij} and $\frac{1}{2}m_n^{Ii}$ are required to be even to ensure the \mathbb{Z}_2 quotient of the $K3$ fibration over T^2 remains well defined as a fibration over $\mathbb{P}^1 = T^2/\mathbb{Z}_2$. This will become more transparent in section 6. There, we will see that the monodromy about each of the four fixed points on \mathbb{P}^1 is $\Gamma_5(\pm\frac{1}{2})\Gamma_6(\pm\frac{1}{2})$, where the two \pm signs are uncorrelated.¹¹ On \mathbb{P}^1 , the monodromies about the x^5 and x^6 circles of T^2 becomes monodromies about *pairs* of fixed points.

The twisted closure relation (3.28) is expressed in terms of ζ_a and $dx^n \wedge \zeta_a$, but can be re-expressed in terms of the ω_a , α_A and β^A of sections 3.1 and 3.2. The twist matrices M_a^A and N_{aA} of eq. (2.3) that result from the nontrivial $K3$ fibration described in this section are then found to be

$$M_a^A = \begin{pmatrix} \gamma_{53}^2 & 0 & \gamma_{63}^1 & 0 \\ \beta_5^{32} & -\gamma_{61}^3 & \beta_6^{31} & -\gamma_{52}^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ m_5^{J2} & 0 & m_6^{J1} & 0 \end{pmatrix}, \quad N_{aA} = \begin{pmatrix} 0 & \gamma_{53}^1 & 0 & -\gamma_{63}^2 \\ \gamma_{62}^3 & \beta_5^{31} & \gamma_{51}^3 & -\beta_6^{32} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & m_5^{J1} & 0 & -m_6^{J2} \end{pmatrix}. \quad (3.34)$$

Here, the ω_a run over ω_α and $\omega_{\alpha I}$, for $\alpha = 1, 2, 3$, and there is an implicit δ_{IJ} lowering the upper J indices in on the m_n^{Ji} . As a check, note that the nonzero entries in the first three rows agree with eq. (3.8). The third row vanishes since ω_3 is Poincaré dual to the generic $K3$ fiber in our construction, which is boundaryless. The fourth and fifth rows vanish, since we incorporate only $\omega_{3I} = \zeta_I$, the blow-up modes of the $K3_{(3)} = T^4_{(3)}/\sigma_3$ fiber, in the twists and not the remaining blow-up modes ω_{1I} and ω_{2I} of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. When we consider a type IIA orientifold based on this geometry in section 4 below, we will require $M_a^A = 0$. In this case, note that an easy way to satisfy the restriction to parabolic class, $(\gamma_n)^2 = 0$, is to set $\gamma_{62}^3 = \gamma_{51}^3 = 0$, so that the nonzero twists are N_{a1} and N_{a3} for $a = 1, 2$ and $3I$.

Fibrations with nonlinear action on $K3$ moduli. The matrix M_p of eq. (3.27) is not the most general Lie algebra element of $\mathfrak{so}(3, 19)$. More general monodromies are possible that do not act linearly on the $K3$ moduli E^λ_i , B^{ij} and A^{Ii} . An arbitrary element of

¹¹For example, in section 6.2, we will see that $\frac{1}{2}m_n^{Ii}$ is required to be even for the monodromies about individual fixed points to lie in the modular group $\Gamma_{3,19}$, which for our parametrization of the $K3$ moduli space contains $A^{Ii} \rightarrow A^{Ii} + 2$ but not $A^{Ii} \rightarrow A^{Ii} + 1$.

$\mathfrak{so}(3, 19)$ can be written

$$M_p = \begin{pmatrix} \gamma_p & \beta_p & m_p^T \\ -h_p & -\gamma_p^T & -n_p^T \\ n_p & -m_p & -f_p \end{pmatrix}. \quad (3.35)$$

Here, the new components compared to eq. (3.27) are n_{pi}^I , h_{pij} (antisymmetric in i, j) and f_p^{IJ} (antisymmetric in I, J). If we use the monodromy matrices $\Gamma_p(x)$ constructed from this general form for M_p to define the global 2-forms ζ_a (cf. eq. (3.24)), then the twisted closure relations (3.28) generalize to

$$\begin{aligned} d\zeta^i &= -dx^p \wedge (\gamma_{pj}^i \zeta^j + \beta_p^{ij} \zeta_j + m_p^{Ii} \zeta_I), \\ d\zeta_i &= dx^p \wedge (h_{pij} \zeta^j + \gamma_{pi}^j \zeta_j + n_{pj}^I \zeta_I), \\ d\zeta_I &= dx^p \wedge \delta_{IJ} (-n_{pi}^J \zeta^I + m_p^{Ji} \zeta_i + f_p^{JK} \zeta_K). \end{aligned} \quad (3.36)$$

For $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2) = (K3_{(3)} \times T_{(3)}^2)/\sigma_1$, the twist components compatible with the \mathbb{Z}_2 involution σ_1 are all of the n_{pi}^I , the components h_{p3i} for $i = 1, 2$, and none of the f_p^{JK} . As we will see in section 5, for other Voisin-Borcea manifolds, the \mathbb{Z}_2 compatible twist components will change, and some of the f_p^{JK} will be retained as well.

When we include the additional data n_{pi}^I and h_{p3i} , the matrices M_a^A and N_{aA} of eq. (3.34) generalize to

$$M_a^A = \begin{pmatrix} \gamma_{53}^2 & -h_{631} & \gamma_{63}^1 & -h_{532} \\ \beta_5^{32} & -\gamma_{61}^3 & \beta_6^{31} & -\gamma_{52}^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ m_5^{J2} & n_{61}^J & m_6^{J1} & n_{52}^J \end{pmatrix}, \quad N_{aA} = \begin{pmatrix} h_{632} & \gamma_{53}^1 & -h_{531} & -\gamma_{63}^2 \\ \gamma_{62}^3 & \beta_5^{31} & \gamma_{51}^3 & -\beta_6^{32} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -n_{62}^J & m_5^{J1} & n_{51}^J & -m_6^{J2} \end{pmatrix}. \quad (3.37)$$

Note that the nonzero entries in the first three rows agree with eq. (3.8), provided that we make the identifications $\beta_p^{ij} = \epsilon^{ijk} \gamma_{pk}^A$ (as discussed above) and $h_{pij} = \epsilon_{ijk} \gamma_{p4}^k$.

In summary, the result of the construction described here — using a nontrivial $K3_{(3)}$ fibration to twist the Calabi-Yau manifold $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ — is twist matrices M_a^A and N_{aA} that are nonzero for $a = 1, 2$ and $3I$, but vanishing for $a = 3, 1I$ and $2I$.

4. The $\mathcal{N} = 1$ type IIA orientifold of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$

Let us consider the $\mathcal{N} = 1$ theory obtained from type IIA string theory on $X = T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ via the orientifold operation described at the end of section 3.1. This is the traditional setting for intersecting D6 brane model building, however, we will focus on the closed string sector here. In this section, our conventions and treatment are very similar to those in ref. [1]. We neglect backreaction in the form of warping and nontrivial dilaton profile. This type of backreaction is irrelevant for the purposes of studying moduli stabilization in supersymmetric vacua. To illustrate this, we restore both the warping and dilaton profile in the first example below.

4.1 Superpotential

The perturbative superpotential is [16, 42, 9, 1, 23, 5]

$$W = W_{\text{NS}} + W_{\text{RR}}, \quad (4.1)$$

where

$$W_{\text{NS}} = - \int_X \Omega_c \wedge (\bar{H} + dJ_c) \quad \text{and} \quad W_{\text{RR}} = \int_X e^{J_c} \wedge \bar{F}_{\text{RR}}. \quad (4.2)$$

Here, \bar{H} and \bar{F}_{RR} denote the background, moduli independent components of the flux only. (This is the same notation as that used in ref. [1]. For a discussion of the closure properties of the various fluxes that appear in this paper, see section 4.3 below.) The quantities J_c and Ω_c are the complexified Kähler form and 3-form appropriate to the orientifold, with lengths measured in units of $(2\pi)^2 \alpha'$,

$$\begin{aligned} J_c &= B + iJ = iT^a \omega_a, & iT^a &= b^a + iv^a, \\ \Omega_c &= C_{(3)} + ie^{-\phi} \text{Re } \Omega = iU^A \alpha_A. \end{aligned} \quad (4.3)$$

Here, B and $C_{(3)}$ are the continuous modulus components of the NS 2-form and RR 3-form potentials, respectively. They do not contribute to the quantized background fluxes \bar{H} and \bar{F}_{RR} . Their components B^a and $C_{(3)}^A$ with respect to the basis forms ω_a and α_A are independent of the internal coordinates. They contribute only continuous moduli dependent contributions to the total flux, do to the lack of closure of the latter:

$$\begin{aligned} dB &= B^a d\omega_a = B^a N_{aA} \beta^A, \\ dC_{(3)} &= C_{(3)}^A d\beta^A = -C_{(3)}^A N_{aA} \tilde{\omega}^a. \end{aligned} \quad (4.4)$$

Note that the β^A component of $C_{(3)}$ is removed by the orientifold projection.

It is convenient to separate the Kähler moduli into T^α , $\alpha = 1, 2, 3$, in the untwisted sector (cf. eq. (3.6)) and $T^{\alpha I}$, $\alpha = 1, 2, 3$, $I = 1, \dots, 16$, in the twisted sector (cf. eq. (3.3)). Likewise, the moduli U^A separate into complex structure moduli U^α and the 4D dilaton-axion $U^0 = S$. Our conventions for J , Ω and the volume form are

$$\frac{i}{8} \Omega \wedge \bar{\Omega} = \frac{1}{6} J \wedge J \wedge J = \text{Vol}_X. \quad (4.5)$$

The complex structure moduli come solely from the untwisted sector of the orbifold. Their geometric components are simply the T^2 complex moduli τ_a , which by the orientifold projection, are required to be purely imaginary, $\tau_\alpha = it_\alpha$. In light of the normalization convention (4.5), the (3,0) form

$$\hat{\Omega} = 2(dx^1 + it_1 dx^2) \wedge (dx^3 + it_2 dx^4) \wedge (dx^5 + it_3 dx^6) \quad (4.6)$$

is related to Ω via

$$\Omega = \left(\frac{V_X}{t_1 t_2 t_3} \right)^{1/2} \hat{\Omega}, \quad (4.7)$$

where $V_X = \int \text{Vol}_X$ is the volume of X . In terms of the α_A and β^A , we have

$$\text{Re } \hat{\Omega} = \alpha_0 + \frac{1}{2} \epsilon^{\alpha\beta\gamma} t_\alpha t_\beta \alpha_\gamma \quad \text{and} \quad \text{Im } \hat{\Omega} = t_\alpha \beta^\alpha + t_1 t_2 t_3 \beta^0. \quad (4.8)$$

The apparent Kähler modulus dependence of Ω_c through the volume dependence (4.7) is an artifact of expressing Ω_c in terms of the 10D rather than 4D dilaton. In terms of the 4D T-duality invariant dilaton $e^{\phi_4} = e^\phi / \sqrt{V_X}$, we have

$$\Omega_c = C_{(3)} + i \text{Re}(C \hat{\Omega}), \quad (4.9)$$

where C is the compensator field

$$C^{-1} = e^{\phi_4} \left(\frac{i}{8} \int \hat{\Omega} \wedge \hat{\Omega} \right)^{1/2} = e^{\phi_4} (t_1 t_2 t_3)^{1/2}. \quad (4.10)$$

The first term in the superpotential comes from NS sector discrete data (NS flux and geometric twists). To evaluate this term, let us write

$$\bar{H} = \bar{H}^A \alpha_A - \bar{H}_A \beta^A, \quad \text{with} \quad \bar{H}^A, \bar{H}_A \in 2\mathbb{Z}. \quad (4.11)$$

For the orientifold, both \bar{H}^A and the geometric twists M_a^A of section 3.1 are projected out. Thus, $dJ_c = iT^a d\omega_a = iT^a N_{aA} \beta^A$, and we obtain

$$W_{\text{NS}}(T, U) = iU^A (\bar{H}_A - iT^a N_{aA}). \quad (4.12)$$

Note that the condition $M_a^A = 0$ implies that $d(\text{Im } \Omega) = 0$, so that the geometry is what called *half flat*.¹²

The second term in the superpotential comes from RR sector discrete data (RR flux). In this case, we write

$$\bar{F}_{(0)} = q^0, \quad \bar{F}_{(2)} = q^a \omega_a, \quad \bar{F}_{(4)} = p_a \tilde{\omega}^a, \quad \bar{F}_{(6)} = p_0 \alpha_0 \wedge \beta^0, \quad (4.13)$$

where¹³ $q^0, q^a, p_a, p_0 \in 2\mathbb{Z}$. Then,

$$W_{\text{RR}}(T) = -q^0 \mathcal{F}(T) - q^a \mathcal{F}_a(T) + p_a iT^a + p_0, \quad (4.14)$$

where $\mathcal{F}(T)$ is the quantum volume of X , with large radius expression

$$-\mathcal{F}(T) = \frac{1}{6} \int J_c \wedge J_c \wedge J_c = \frac{1}{6} \kappa_{abc} iT^a iT^b iT^c, \quad (4.15)$$

and $\mathcal{F}_a = \partial \mathcal{F} / \partial (iT^a)$. The intersection numbers $\kappa_{abc} = \int_X \omega_a \wedge \omega_b \wedge \omega_c$ can be found in appendix B.

¹²When the backreaction on the geometry in the form of nontrivial warping and dilaton profile are included, this condition becomes $d(e^{-\phi/3} \text{Im } \Omega) = 0$, so that the geometry is instead conformally half flat.

¹³We take the integer quantized NS and RR flux components to be even in order to avoid subtleties involving exotic orientifold planes.

At finite radius, $\mathcal{F}(T)$ receives corrections from worldsheet instantons. Note that $\mathcal{F}(T)$ is not quite the same as the prepotential for the Kähler moduli in the parent $\mathcal{N} = 2$ theory, due to corrections from unoriented worldsheets. The former has been computed for the Voisin-Borcea class (along with the topological string amplitudes for higher genus) [22], but the latter are still unknown.

In addition to worldsheet instanton corrections, the superpotential receives D-instanton corrections from Euclidean D2 branes wrapping generalized special Lagrangian 3-cycles [7]. The 1-instanton contribution from a D2 brane wrapping the 3-cycle $C_A A^A$ takes the form $\text{Pfaff}(T)e^{-2\pi C_A U^A}$, where $\text{Pfaff}(T)$ is the 1-loop Pfaffian prefactor. These are the mirrors of the D3 instantons in KKLТ [31] type IIB backgrounds. Similarly, from gaugino condensation on stacks of D6 branes wrapping the 3-cycle $C_A A^A$ one can obtain corrections of fractional D2 instanton number. This is the mirror of gaugino condensation on D7 brane stacks in type IIB.

4.2 Kähler potential

The Kähler potential is $K = K_1(T, \bar{T}) + K_2(U, \bar{U})$, where at large radius [1, 23, 5]

$$\begin{aligned}
 K_1(T, \bar{T}) &= -\log \frac{4}{3} \int_X J \wedge J \wedge J = -\log \frac{1}{6} \kappa_{abc} (T^a + \bar{T}^a)(T^b + \bar{T}^b)(T^c + \bar{T}^c), \\
 K_2(U, \bar{U}) &= -2 \log \frac{i}{2} \int_X C \hat{\Omega} \wedge \bar{C} \hat{\Omega} = -\sum_{\alpha=0}^3 \log(U^\alpha + \bar{U}^\alpha),
 \end{aligned}
 \tag{4.16}$$

for the Kähler moduli and the combined dilaton-axion/complex structure moduli, respectively. At finite radius, the Kähler potential receives corrections from both worldsheet instantons and D-brane instantons.

4.3 Bianchi identities

The only nontrivial Bianchi identity (tadpole cancellation condition) for the background flux is that for $\bar{F}_{(2)}$:

$$d\bar{F}_{(2)} = \bar{H} \wedge \bar{F}_{(0)} + j_{D6,O6}, \tag{4.17}$$

where $j_{D6,O6}$ is the source term due to D6 branes and O6 planes.

Let us write

$$H = \bar{H} + dB, \quad F_{(4)} = \bar{F}_{(4)} + dC_{(3)}, \tag{4.18}$$

and $\bar{F}_{(n)} = F_{(n)}$ for $n = 0, 2, 6$. In addition, let us define $\tilde{F}_{RR} = e^B F_{RR}$. Note that, in contrast to eq. (4.17), the \tilde{F} satisfy the modified Bianchi identities with total (background plus moduli dependent) flux,

$$d\tilde{F}_{(8-p)} = H \wedge F_{(6-p)} + j_{Dp,Op}. \tag{4.19}$$

The conditions (4.19) constrain the moduli B and $C_{(3)}$, however we need not concern ourselves with these non-topological constraints here. These constraints are automatically

taken care of by the supersymmetry conditions below. For later reference, the tilded fluxes appearing in the supersymmetry conditions are

$$\tilde{F}_{(2)} = \bar{F}_{(2)} + B \wedge \bar{F}_{(0)}, \tag{4.20}$$

$$\tilde{F}_{(4)} = \bar{F}_{(4)} + dC_{(3)} + B \wedge F_{(2)} + \frac{1}{2}B \wedge B\bar{F}_{(0)}, \tag{4.21}$$

and $\tilde{F}_{(6)}$, whose precise decomposition we will not need.

Now let us return to the topological Bianchi identity (4.17). Using eq. (4.13), together with the relations

$$\begin{aligned} A^B \cap B_A &= \int_{A^B} \alpha_A = \int_X \alpha_A \wedge \beta^B = \delta_A^B, \\ B_A \cap A^B &= \int_{B_A} \beta^B = \int_X \beta^B \wedge \alpha_A = -\delta_A^B \end{aligned} \tag{4.22}$$

(i.e., A^B, B_A Poincaré dual to β^B, α_A , respectively), we obtain

$$-(q^0 \bar{H}_A + q^a N_{aA})A^A + \sum_{\nu} N^{\nu}(\pi_{\nu} + \pi'_{\nu}) = 4\pi_{O6}. \tag{4.23}$$

Here, the sum runs over stacks of N^{ν} D6 branes wrapping homology class π_{ν} and N^{ν} image D6 branes wrapping homology class π'_{ν} . Writing, as in ref. [8],

$$\begin{aligned} \pi_{\nu} &= \sum_{A=0}^3 (X_{\nu A} A^A + Y_{\nu}^A B_A), \\ \pi'_{\nu} &= \sum_{A=0}^3 (X_{\nu A} A^A - Y_{\nu}^A B_A), \\ \pi_{O6} &= 4 \sum_{A=0}^3 A^A, \end{aligned} \tag{4.24}$$

the Bianchi identities become¹⁴

$$-\frac{1}{2}(q^0 \bar{H}_A + q^a N_{aA}) + \sum_{\nu} N^{\nu} X_{\nu A} = 8, \quad \text{for } A = 0, 1, 2, 3. \tag{4.25}$$

In the next section, we will discuss supersymmetric Minkowski vacua, for which $q^0 = 0$ and $d(e^{-\phi} \text{Re } \Omega) = \star \bar{F}_{(2)}$. In this case,

$$0 \leq \int_X \bar{F}_{(2)} \wedge \star \bar{F}_{(2)} = \int_X \bar{F}_{(2)} \wedge d(e^{-\phi} \text{Re } \Omega) = -(q^a N_{aA}) \text{Re } U^A. \tag{4.26}$$

The geometric regime is $\text{Re } U^A > 0$. In the case that the solutions have a moduli space in which all of the $\text{Re } U^A$ can be varied independently, it follows that the contribution $-\frac{1}{2}(q^0 \bar{H}_A + q^a N_{aA})$ to the Bianchi identity from discrete data is nonnegative. We suspect that this is true in general for supersymmetric Minkowski vacua, irrespective of the resulting moduli space. (For AdS vacua, see ref. [9].)

¹⁴The change in sign of the relative contribution to π_{O6} from $A = 0$ and $A = 1, 2, 3$ as compared to refs. [1] and [8], comes from the different convention used in this paper for the α_A, β^A . In the absence of geometric twists, these Bianchi identities were first derived for $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ in ref. [11].

4.4 Supersymmetric Minkowski vacua

The scalar potential from F -terms in $\mathcal{N} = 1$ supergravity is

$$\mathcal{V}(\Phi, \bar{\Phi}) = e^K \left(\sum_{P,Q} K^{P\bar{Q}} D_P W(\Phi) \bar{D}_{\bar{Q}} \bar{W}(\bar{\Phi}) - 3 \left| W(\Phi) \right|^2 \right), \quad (4.27)$$

where $D_P = \partial_{\Phi^P} - K_{,P}{}^{\bar{Q}} D_{\bar{Q}}$ for each modulus Φ^P . Supersymmetric vacua satisfy $D_P W = 0$ for all Φ^P . Thus, $V = -3e^K |W|^2$, which generically gives Anti de Sitter vacua. However, in the special case $W = 0$, we obtain Minkowski vacua. Let us focus on this case. Then the supersymmetry conditions become considerably more manageable, in that they become truly holomorphic equations $\partial_{\Phi^P} W(\Phi) = 0$, as opposed to just covariantly holomorphic.

The holomorphic monopole equations. From the superpotential (4.1), the supersymmetry conditions that we obtain in this way are

$$d\Omega_c + (e^{J_c} \wedge \bar{F}_{RR})_{(4)} = 0, \quad \bar{H} + dJ_c = 0. \quad (4.28)$$

The imaginary and real parts are

$$d(e^{-\phi} \text{Re } \Omega) + J \wedge \tilde{F}_{(2)} = 0, \quad dJ = 0, \quad (4.29)$$

and

$$\tilde{F}_{(4)} - \frac{1}{2} J \wedge J \bar{F}_{(0)} = 0, \quad H = 0, \quad (4.30)$$

respectively. In addition, there is the Minkowski condition $W = 0$. When combined with eq. (4.30) and (4.29), the Minkowski condition implies that the $SU(3)$ singlet components of the tilded fluxes vanish,¹⁵

$$\tilde{F}_{(6)} = J \wedge \tilde{F}_{(4)} = J \wedge J \wedge \tilde{F}_{(2)} = J \wedge J \wedge J F_{(0)} = 0. \quad (4.31)$$

Thus, the supersymmetry conditions reduce to

$$d(e^{-\phi} \text{Re } \Omega) + J \wedge \bar{F}_{(2)} = 0, \quad dJ = 0, \quad \bar{F}_{(2)} \text{ primitive}, \quad (4.32)$$

with $\tilde{F}_{(6)} = \tilde{F}_{(4)} = F_{(0)} = 0$. Here, $\bar{F}_{(2)}$ primitive means that $\bar{F}_{(2)} \wedge J \wedge J = 0$. These are the *holomorphic monopole* equations discussed in refs. [34, 20]. As already mentioned in a footnote above, one can also show that the imaginary part of Ω satisfies $d(e^{-\phi/3} \text{Im } \Omega) = 0$.

Note that the first condition in eq. (4.32) is a generalized calibration condition. Since $\star \bar{F}_{(2)} = -J \wedge \bar{F}_{(2)}$ for $\bar{F}_{(2)}$ primitive, it can be rewritten as

$$d(e^{-\phi} \text{Re } \Omega) = \star \bar{F}_{(2)}. \quad (4.33)$$

The equation relates the generalized calibration $e^{-\phi} \text{Re } \Omega$, which calibrates (serves as the volume form on) the generalized special Lagrangian cycles on which we can wrap D6 branes, to the flux $\bar{F}_{(2)}$, which is sourced by D6 branes.

¹⁵This involves observing that the same $SU(3)$ singlet torsion component W_1 appears in both $dJ = -\frac{3}{2} \text{Im}(W_1 \bar{\Omega}) + W_4 \wedge J + W_3$ and $d\Omega = W_1 J^2 + W_2 \wedge J + \bar{W}_5 \wedge \Omega$, and that $W_1 = 0$ from the second equation in (4.29).

The holomorphic monopole background is the Kaluza-Klein reduction of M theory on $\mathbb{R}^{3,1}$ times a manifold of G_2 holonomy, using the the M theory circle for the reduction. The lift to a M theory on G_2 is guaranteed on general grounds by $\mathcal{N} = 1$ supersymmetry and the fact that the $\tilde{F}_{(2)}$ flux, D6 branes and O6 planes each have purely geometrical M theory lifts. Due to the O6 planes in the IIA background, the M theory geometry does not actually have a circle fiber with a $U(1)$ isometry on which to dimensionally reduce.¹⁶ As discussed in refs. [39, 40], the reduction involves first truncating to the lowest Kaluza-Klein modes of an approximate $U(1)$ isometry that fails only locally near the regions of the M theory geometry that become the O6 planes in IIA. The neglected M theory corrections from higher Kaluza-Klein modes are corrections from D0 bound states in IIA. These corrections are appreciable only near the O6 planes, where the string coupling diverges and the D0 bound states become light.

Although the conclusion is that we have rediscovered the class of M theory compactifications on G_2 holonomy manifolds, the description in terms of the $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orientifold with well defined discrete data (D6 branes, $\tilde{F}_{(2)}$ flux and geometric twists) should provide a useful route to studying these $\mathcal{N} = 1$ compactifications for model building purposes that is more explicit than that which currently exists from the G_2 perspective.

Simplifying assumptions. In light of the holomorphic monopole equations obtained in the previous section, we set $\bar{H} = \bar{F}_{(6)} = \bar{F}_{(4)} = 0$ for simplicity from now on. For choices of discrete data that are compatible with the supersymmetry conditions, giving nonzero values to these background fluxes just leads to compensatingly shifted expectation values for some of the axionic moduli, to ensure that the corresponding tilded fluxes vanish, as required by the supersymmetry conditions. This simplifying assumption can be thought of a discrete gauge choice. Note that since $q^0 \equiv \bar{F}_{(0)} = 0$ for Minkowski vacua, \bar{H}_A disappears from the quantity $-\frac{1}{2}(q^0 \bar{H}_A + q^a N_{aA})$ that appears in the RR tadpole cancellation conditions.

General observations and validity of the classical supergravity analysis. Classically, the conditions $\partial_P W = 0$ and $W = 0$ are homogeneous of degree 1 and 2 respectively under rescaling of all moduli $(T^a, U^A) \rightarrow (\lambda T^a, \lambda U^A)$. This means that for generic choice of discrete data q^a, N_{aA} , and neglecting worldsheet and D2 instanton corrections, the moduli T^a and U^A all run away to zero. In this case, the instanton corrections will be critical in understanding the stabilization of moduli. On the other hand, for nongeneric discrete data, we can classically have a nontrivial moduli space in which ratios of moduli are fixed but the overall scale is not. Recall that the complex structure moduli are actually the inhomogeneous coordinates U^A/U^0 , where $U^0 = S$ is the dilaton-axion. Fixing ratios of T^a, U^A but not the overall scale means that we can fix complex structure moduli and relative Kähler moduli and/or their ratios, as well as the ratio of the overall volume modulus to the dilaton-axion S . However, the dilaton-axion is left freely tunable. At large S , we are in the weak coupling regime and the large volume regime for the nonzero Kähler moduli. One can then hope to simultaneously break supersymmetry and lift the dilaton and any re-

¹⁶Indeed, there is a theorem that a compact manifolds of special holonomy cannot have continuous isometries.

maintaining complex structure moduli by including anti-D6 branes and D2 instantons (and/or gaugino condensation), in a construction mirror to that of KKL'T [31].

Note that if any of the Kähler moduli vanishes, then we are clearly in a regime where the large volume classical supergravity description is not valid and worldsheet instanton corrections are critical. We will not have much to say quantitatively about this regime here. However, note that for orbifolds, the extreme opposite of the large volume regime is also computationally accessible in the following sense. When all moduli in the orbifold untwisted sector are large, there is a perturbative expansion about the orbifold limit, in powers of the small blow-up moduli. (See, for example, ref. [10].) This will be relevant for putting our examples of Minkowski vacua on firmer footing. In three of the examples we present below, we will find that that we are in exactly this situation — in the classical supergravity analysis, all Kähler moduli in the orbifold untwisted sector of $(K3 \times T^2)/\mathbb{Z}_2$ can be taken large, while those in the orbifold twisted sector are required to vanish. We leave the requisite conformal field theory analysis about the orbifold limit for future work.

Finally, note that worldsheet instanton corrections are also important when the Kähler moduli are of order α' . We will never be forced to this regime for the Minkowski vacua discussed here, due to the homogeneity property. However, it is generically the case for the vacua discussed in ref. [1]. There, nongeometric twists are included as well, so the superpotential is no longer homogeneous, and the supersymmetry condition become polynomial equations whose roots are generically of order 1 in α' units. Therefore, the analysis of ref. [1] is technically only applicable when viewed as the truncation of the $\mathcal{N} = 4$ orientifold on T^6 to the diagonal $T^2 \times T^2 \times T^2$. In this case, the degree of supersymmetry protects the theory against worldsheet instantons, but the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is subject to order 1 corrections.

Example 1: twists inherited from T^6 only. This example includes $\bar{F}_{(2)}$ flux and geometric twists in the orbifold untwisted sector of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ only. It illustrates two important features of the Minkowski vacua of the IIA orientifold. First, when the blow-up moduli are included in the analysis, they can nonetheless still be fixed due to the interplay of twisted and untwisted sector in the intersection numbers of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, which enter into the superpotential contribution $W_{RR}(T, U)$. Second, we claimed above that the backreaction (warping) of the geometry and nontrivial dilaton due to the presence of the D6 branes and O6 planes, were irrelevant for the purposes of studying moduli stabilization in supersymmetric vacua. In this example, we restore the warp factors Z and dilaton profile to illustrate this point.

For simplicity, let us solve the Bianchi identities (4.25) for $A = 0, 1, 2$, by locally cancelling the O6 charge with coincident D6 branes. For $A = 3$, we include $N = 8 - 4qn$ individual D6 branes at arbitrary positions $x_\nu^1, x_\nu^3, x_\nu^6$ for $\nu = 1, \dots, 8 - 4qn$. (The integers q and n will be defined below.)

On the $\mathbb{R}^{3,1} \times$ twisted- T^6 covering space, the geometry is

$$ds^2 = Z^{-1/2}(\eta_{\mu\nu}dx^\mu dx^\nu + ds_{\parallel}^2) + Z^{1/2}ds_{\perp}^2 \tag{4.34}$$

where

$$\begin{aligned} ds_{\parallel}^2 &= (R_2\eta^2)^2 + (R_4\eta^4)^2 + (R_5dx^5)^2, \\ ds_{\perp}^2 &= (R_1dx^1)^2 + (R_3dx^3)^2 + (R_6dx^6)^2, \end{aligned} \tag{4.35}$$

are the metrics on the T^3 factors parallel and perpendicular to the worldvolumes of the D6 branes that wrap cycles of homology class A^3 . The $(1,0)$ -forms are spanned by the complex vielbein

$$\begin{aligned} \eta^{z^1} &= Z^{1/4}R_1dx^1 + iZ^{-1/4}R_2\eta^2, \\ \eta^{z^2} &= Z^{1/4}R_3dx^3 + iZ^{-1/4}R_4\eta^4, \\ \eta^{z^3} &= Z^{-1/4}R_5dx^5 + iZ^{1/4}R_6dx^6. \end{aligned} \tag{4.36}$$

Then, on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, if we set the blow-up moduli to zero, the fundamental form¹⁷ and $(3,0)$ form are

$$\begin{aligned} J &= 2v^1dx^1 \wedge \eta^2 + 2v^2dx^3 \wedge \eta^4 + 2v^3dx^5 \wedge dx^6, \\ \Omega &= 2\eta^{z^1} \wedge \eta^{z^2} \wedge \eta^{z^3}, \end{aligned} \tag{4.37}$$

where $v_1 = R_1R_2$, $v_2 = R_3R_4$ and $v_3 = R_5R_6$.

Let us choose T^6 twists

$$d\eta^2 = -2ndx^3 \wedge dx^6 \quad \text{and} \quad d\eta^4 = -2ndx^1 \wedge dx^6. \tag{4.38}$$

This gives twist data $\gamma_{63}^2 = -2n$ and $\gamma_{61}^4 (= -\beta_6^{32}) = -2n$, so that eq. (3.8) becomes

$$N_{aA} = \begin{pmatrix} 0 & 0 & 0 & 2n \\ 0 & 0 & 0 & -2n \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.39}$$

In the RR sector, we take

$$\bar{F}_{(2)} = g_s^{-1} \star_3 dZ - 4q(dx^1 \wedge \eta^2 - dx^3 \wedge \eta^4). \tag{4.40}$$

Here, the first term is the backreaction from the D6 branes and O6 planes, with \star_3 the Hodge star operator in the metric ds_{\perp}^2 . The second term gives the discrete RR data $q^1 = -2q$ and $q^2 = 2q$.

The dilaton acquires a nontrivial profile in the extra dimension. We have $e^{\phi} = g_s Z^{-3/4}$. The warp factor satisfies

$$-\nabla_{\perp}^2 Z = g_s \left(N_3^{\text{discrete}} + \sum_{\text{D6,O6}} Q_{\text{D6,O6}} \delta^3(x - x_{\text{D6,O6}}) \right) / \sqrt{g_{\perp}}, \tag{4.41}$$

where

$$N_A^{\text{discrete}} = -2(q^0 \bar{H}_A + q^a N_{aA}). \tag{4.42}$$

¹⁷When the fundamental form J is closed it is called the Kähler form.

Here, $Q_{D6} = 1$ and $Q_{O6} = -4$. The sum runs over all O6 planes, D6 branes and image D6 branes that wrap cycles in T^6 which project to the class A^3 in $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. On the T^6 covering space (with respect to both the orientifold and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold), each of these D6 branes appears as part of a quadruple, consisting of one D6 brane plus three image D6 branes.¹⁸ In addition, there are 8 parallel O6 planes, located at the points on the transverse space $T^3_\perp(x^1, x^3, x^6)$ where each coordinate is equal to 0 or 1/2. The total source charge on the right hand side of eq. (4.41) is zero, as required in order to solve Poisson's equation on a compact manifold. The contributions are $N_3^{\text{discrete}} = 16qn$, $\sum Q_{O6} = -32$ from eight O6 planes, and $\sum Q_{D6} = 4(8 - 4qn)$ from $(8 - 4qn)$ quadruples of D6 branes plus images.

One can check that this background indeed satisfies the holomorphic monopole equations, provided that the moduli satisfy the constraints $T^1 = T^2$ and $U^3 = (2q/n)T^3$ derived below. In particular, the equations

$$d(e^{-\phi} \text{Re} \Omega) + J \wedge \bar{F}_{(2)} = 0, \quad d(e^{-\phi/3} \text{Im} \Omega) = 0, \quad dJ = 0,$$

are satisfied, even when one includes the nontrivial dilaton profile and warping of the geometry.

We now derive the moduli constraints using the superpotential $W = W_{\text{NS}} + W_{\text{RR}}$. From the matrix N_{aA} above, the NS superpotential is

$$W_{\text{NS}} = 2nU^3(T^1 - T^2). \tag{4.43}$$

The RR superpotential is

$$W_{\text{RR}} = 2q\mathcal{F}_1 - 2q\mathcal{F}_2. \tag{4.44}$$

Let us first neglect the blow-up modes. Then,

$$\begin{aligned} W_{\text{RR}} &= \left(2q \frac{\partial}{\partial(iT^1)} - 2q \frac{\partial}{\partial(iT^2)} \right) (-2iT^1 iT^2 iT^3) \\ &= -4q(T^1 - T^2)T^3. \end{aligned} \tag{4.45}$$

So, the total superpotential is

$$W = W_{\text{NS}} + W_{\text{RR}} = (2nU^3 - 4qT^3)(T^1 - T^2). \tag{4.46}$$

The supersymmetry conditions are $T^1 = T^2$ from $\partial_T W = 0$ and $U^3 = (2q/n)T^3$ from $\partial_U W = 0$.

We can also obtain these constraints by duality as follows. This background (without the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold and D6/O6 stacks for $A = 0, 1, 2$) was studied in ref. [40] as in intermediate background in the duality chain between the type IIB orientifold $T^6/\Omega(-1)^{F_L} \mathcal{I}_6$ [32]

¹⁸This is somewhat counterintuitive. One might have expected the the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold introduces a factor of 4 in the number of D6 branes plus images, and then the orientifold introduces another factor of 2, for a factor of 8 rather than 4 total. This is not the way it works. Since the D6 branes wrap SLAG cycles, while the orbifold $\mathbb{Z}_2 \times \mathbb{Z}_2$ inverts holomorphic cycles, one finds that orbifold doubles rather than quadruples the number of branes needed on the covering space. For $2N$ parallel D6 branes on the covering spaces, one finds $U(N) \rightarrow U(N/2) \times U(N/2) \rightarrow U(N/2)$, upon implementing the first and then second orbifold \mathbb{Z}_2 projection. So indeed, the rank decreases by a factor of 2 rather than 4 from the orbifold $\mathbb{Z}_2 \times \mathbb{Z}_2$.

and a type IIA Calabi-Yau dual. In the IIB orientifold, the first constraint is the condition $\det_{2 \times 2} \tau = -1$ on a T^4 factor and the second is $\tau_3 \tau_{\text{dil}} = -1$ on a T^2 factor. These are the type IIB supersymmetry conditions that the complex 3-form flux be (2,1) and primitive.¹⁹

If we include the blow-up modes, then using the classical intersection matrix for $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ given in appendix B, the new RR superpotential is

$$W_{\text{RR}} = 4q \left(T^2 T^3 - \sum_I (T^{1I})^2 \right) - 4q \left(T^1 T^3 - \sum_I (T^{2I})^2 \right). \quad (4.47)$$

Varying with respect to the blow-up Kähler moduli $T^{\alpha I}$, we obtain $T^{1I} = T^{2I} = 0$ and T^{3I} arbitrary. Then, varying with respect to T^α and U^A gives the same constraints as before.

This example shows that even when we have not included twists or RR data that involve the blow-up modes of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, the blow-up moduli can nevertheless be stabilized due to the interplay of orbifold twisted sector and untwisted sector in the intersection form.

Of course, our large radius analysis is not reliable at $T^{1I} = T^{2I} = 0$ and we should instead use the appropriate $\mathcal{F}(T)$ for a very small resolution of the orbifold. However, all is not lost, since the latter complementary regime is also computationally accessible from the orbifold conformal field theory for $(K3 \times T^2)/\mathbb{Z}_2$, as already mentioned above. We will not need the full computation of $\mathcal{F}(T)$, but can make do with a pair of results analogous to those proven in ref. [10] in the slightly different context of the heterotic \mathbb{Z}_3 orbifold. Assuming that these results carry over here as well, we would find the following: $\partial^2 \mathcal{F} / \partial T^{\text{ut}} \partial T^{\text{tw}}$ vanishes at $T^{\text{tw}} = 0$ and $\partial^2 \mathcal{F} / \partial T^{\text{ut}} \partial T^{\text{ut}}$ gives the classical result at $T^{\text{tw}} = 0$ and large T^{ut} , where T^{ut} and T^{tw} are the orbifold untwisted and twisted sector Kähler moduli, respectively. This would imply that the superpotential extrema obtained above at $T^{1I} = T^{2I} = 0$ persist in the presence of corrections from worldsheet instantons. For now we leave the problem open, but we hope to return to the question of the validity of this result in future work.

Example 2: The shrinking K3 surface. The additional discrete data at our disposal that was not used in the previous example is the blow-up mode twist data $N_{A,3I}$, and the RR flux data q^3 and q^{3I} . The real complication of the supersymmetry conditions comes from the intersection form of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, which appears in W_{RR} . Therefore, as a first step toward generalization, let us attempt to find supersymmetric Minkowski vacua in which the q^{3I} still vanish. The complete solution can be described as follows.

From $\partial W / \partial T^\alpha = 0$, we have

$$T^1 = \frac{1}{2q^2 q^3} (A - B), \quad T^2 = -\frac{1}{2q^1 q^3} (A - B), \quad T^3 = -\frac{1}{2q^1 q^2} (A + B), \quad (4.48)$$

where $A = -\frac{1}{2} q^1 N_{1A} U^A$ and $B = -\frac{1}{2} q^2 N_{2A} U^A$. For the geometric regime $\text{Re} T^\alpha > 0$, we take q^1 negative, q^2, q^3 positive, and require that $A > B > 0$. From $\partial W / \partial T^{\alpha I} = 0$, we find

$$T^{1I} = T^{2I} = 0 \quad \text{and} \quad T^{3I} = -\frac{1}{2q^3} N_{3I,A} U^A. \quad (4.49)$$

¹⁹For comparison to ref. [40], note that $2q$ here equals m there, and that the (1, 2, 3, 4, 5, 6) directions here are the (6, 5', 7, 4', 9', -8) directions there. For the moduli constraints, the correspondence is that $T^1 = T^2$, $(2q/n)T^3 = U^3$ here is $\tau'_1 = \tau_2$, $g'_s R_s = (n/m)v'_1$ of ref. [40].

So, we are again forced to the $(K3 \times T^2)/\mathbb{Z}_2$ orbifold limit. Finally, from $\partial W/\partial U^A = 0$, we have $\Pi_{AB}U^B = 0$, where

$$\Pi_{AB} = \frac{1}{4|q^1q^2q^3|} \left((q^1N_{1A} - q^2N_{2A})(q^1N_{1B} - q^2N_{2B}) - 2|q^1q^2| \sum_I N_{3I,A}N_{3I,B} \right). \quad (4.50)$$

For compatibility of this last condition with the geometric regime $\text{Re } U^A > 0$ and $A > B > 0$, the N_{aA} must be chosen in such a way that $\Pi_{AB} = 0$. The condition $W = 0$ is then automatically satisfied.

However, note the following pathology of this solution. The volume of the $K3$ fiber, $V(K3_{(3)}) = 2T^1T^2 - \sum_I (T^{3I})^2$, vanishes. This can be seen explicitly from eqs. (4.48) through (4.50) above. It can also be seen more directly as follows. For any supersymmetric Minkowski vacuum, we can show that not only does W vanish, but W_{RR} and W_{NS} each vanishes separately. In the present example, $\partial W/\partial T^3 = 0$ gives $2q^1T^2 + 2q^2T^1 = 0$. Using only this equation, $W_{\text{RR}} = -q^3V(K3_{(3)})$.

This is roughly analogous to the result that for type IIB vacua, including 3-form flux through an A -cycle, and no flux through the corresponding B -cycle, forces the geometry to a conifold point in which the B -cycle shrinks to zero size. In our example, we have chosen RR flux q^3 through the 2-cycle Poincaré dual to $\tilde{\omega}^3$, without geometric flux $d(e^{-\phi} \text{Re } \Omega) = -N_{aA}U^A\tilde{\omega}^a$ through the dual 4-cycle $K3_{(3)}$ (which is Poincaré dual to ω_3). In another context, this would perhaps lead to a Calabi-Yau singularity in which a 4-cycle locally shrinks (to i.e., a del Pezzo singularity), but here that cannot happen, so the whole $K3$ fiber shrinks. Of course, this example is well outside the regime of validity of the classical supergravity. It was provided for heuristic purposes only.

Example 3: twists involving $K3$ blow-up modes. Given the lesson learned in the previous example, we can hope to find a supersymmetric Minkowski vacuum without shrinking divisors, by permitting only q^1, q^2, q^{3I} and the dual topological data $N_{1A}, N_{2A}, N_{3I,A}$ to be nonzero. For simplicity, we take the blow-up mode data to be nonzero only for $a = 3I|_{I=1}$, and for notational convenience, write $a = 4$ to denote $a = 3I|_{I=1}$. In order to remain in the parabolic case discussed in section 3.3, we set $N_{aA} = 0$ unless $A = 1, 3$.

From $\partial_a W = 0$ together with $W_{\text{RR}} = 0$, we find that the $(K3 \times T^2)/\mathbb{Z}_2$ blow-up moduli vanish ($T^{1I} = T^{2I} = 0$), all of the T^{3I} are arbitrary, and T^1, T^2, T^3 are constrained as

$$\begin{aligned} T^3 &= \frac{1}{2q^2}N_{1A}U^A = \frac{1}{2q^1}N_{2A}U^A = -\frac{1}{2q^4}N_{4A}U^A, \\ q^2T^1 + q^1T^2 &= q^4T^4. \end{aligned} \quad (4.51)$$

Then, from $\partial_A W = 0$, we have

$$N_{1A}T^1 + N_{2A}T^2 + N_{4A}T^4 = 0. \quad (4.52)$$

Comparing the last set of equations to the second line of eq. (4.51), we see that solutions exist for

$$(N_{1A}, N_{2A}, N_{4A}) = \lambda n_A(q^2, q^1, -q^4). \quad (4.53)$$

Here, n_1 and n_3 can be independently chosen to be 0 or 1, and λ is a proportionality constant. (We take $n_0 = n_2 = 0$ to remain in case of parabolic twists as mentioned in the first paragraph.) Using eq. (4.53), the first line of eq. (4.51) becomes

$$T^3 = \frac{1}{2}\lambda n_A U^A = \frac{1}{2}\lambda(n_1 U^1 + n_3 U^3). \tag{4.54}$$

Let us write $\mathbf{q} = (-q^1, q^2, q^4)^T$. Then two classes of solutions with $\lambda = 1$, and their associated contributions to the D6 Bianchi identities, are

$$\begin{aligned} \mathbf{q}^T &= 2(-1 + z^2, 2, 2z), \quad z \in \mathbb{Z}, \quad -\frac{1}{2}q^a N_{aA} = 8n_A, \\ \mathbf{q}^T &= 2\left(-1 + \frac{1}{2}z^2, 1, z\right), \quad z \in 2\mathbb{Z}, \quad -\frac{1}{2}q^a N_{aA} = 4n_A. \end{aligned} \tag{4.55}$$

Additional examples with $\lambda = \frac{1}{2}$ or 2 are obtained from the second class, by doubling \mathbf{q} only or N_{aA} only, respectively. In both cases, $-\frac{1}{2}q^a N_{aA} = 8n_A$. The number of D6 branes wrapping 3-cycles of homology class A^A is $8 + \frac{1}{2}q^a N_{aA}$. This is either 0, 4, or 8 for each of the classes of solutions that we have just described.

Note that from the $K3_{(3)}$ volume form, which is minus $(-T^1)T^2 + T^2(-T^1) + \sum_I (T^{3I})^2$, we obtain a natural $SO(1, 1 + 16)$ structure on these classes of solutions: the metric is of the form (3.12), and $(-T^1, T^2, T^{3I})^T$ and $(-q^1, q^2, q^{3I})^T$ transform as vectors. (This is the $SO(1, 1 + 16)$ obtained from the $SO(3, 3 + 16)$ coset moduli space of $K3$ by restricting to Kähler deformations only.) Let us restrict to the $SO(1, 1 + 1)$ acting on $(-T^1, T^2, T^4)^T$ and \mathbf{q} . Then the classes above follow from the $z = 0$ class by $SO(1, 1 + 1)$ rotation. The matrix that implements the rotation is

$$v = \begin{pmatrix} 1 & -\frac{1}{2}z^2 & -z \\ 0 & 1 & 0 \\ 0 & z & 0 \end{pmatrix}. \tag{4.56}$$

If we identify solutions that are related by action of the $K3$ modular group (i.e., the action of $SO(1, 2; \mathbb{Z}) \subset \Gamma_{3,19}$), then the first class collapses to just two solutions, corresponding to $z = 0$ and $1 \pmod{2}$. The second class collapses to one solution. The $z = 0$ solutions are analogous to Example 1, with twists inherited from T^6 only. The only difference compared to Example 1 is that we permit q^1 and q^2 to be distinct, and we allow N_{aA} to be nonzero not only for $A = 3$, but for $A = 1$ as well.

Example 4: elliptic twists, but no flux. Consider the choice of discrete data $q^a = 0$, $-N_{11} = -N_{12} = N_{21} = N_{22} = \pi$ and $N_{aA} = 0$ otherwise. From eq. (3.37), this choice corresponds to the $K3$ fibration data

$$\begin{aligned} -\gamma_{53}^1 &= \pi, \\ \beta_5^{31} &= -\beta_5^{13} = -\pi, \\ h_{531} &= -h_{513} = -\pi. \end{aligned} \tag{4.57}$$

Let us take a moment to describe the monodromies in this elliptic case explicitly. First, let I and S denote the 2×2 matrices

$$I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}. \quad (4.58)$$

Then, the twist matrices (3.35) corresponding to the choice (4.57) of β, γ, h are $M_6 = 0$ and

$$M_5 = \pi \begin{pmatrix} S & -S \\ -S & S \end{pmatrix}, \quad (4.59)$$

where we have restricted to the nonzero $\text{SO}(2, 2)$ block of M_5 with indices $i = 1, 3$.

In the identification $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \cong \text{SO}(2, 2; \mathbb{R})$, we have

$$I \otimes S \cong \begin{pmatrix} S & \\ & S \end{pmatrix}, \quad S \otimes I \cong \begin{pmatrix} & S \\ S & \end{pmatrix}, \quad S \otimes S \cong \begin{pmatrix} & -I \\ -I & \end{pmatrix}. \quad (4.60)$$

So, we can write

$$M_5 \cong \pi(I \otimes S - S \otimes I). \quad (4.61)$$

Using $S^2 = -1$, the matrix $\Gamma_5(x^5) = \exp(-M_5 x^5)$ appearing in the definition of the twisted 2-forms (3.24) is

$$\begin{aligned} \Gamma_5(x^5) &\cong \exp(-\pi x^5 I \otimes S) \exp(\pi x^5 S \otimes I) \\ &= \exp(\pi x^5 S) \otimes \exp(-\pi x^5 S) \\ &= (\cos \pi x^5 + S \sin \pi x^5) \otimes (\cos \pi x^5 - S \sin \pi x^5). \end{aligned} \quad (4.62)$$

The monodromy about the x^5 circle is $\Gamma_5(1)$, while that about each of the four fixed points of $\mathbb{P}^1 = T^2(x^5, x^6)/\mathbb{Z}_2$ is $\Gamma(\pm \frac{1}{2})$ (cf. the discussions at the end of section 3.3 and in section 6.1). From the last result, we have

$$\Gamma_5(\pm \frac{1}{2}) \cong -S \otimes S, \quad \Gamma_5(1) = 1. \quad (4.63)$$

Since M_6 vanishes, we have $\Gamma_6(x^6) = 1$.

We now turn to moduli stabilization, as in the previous examples. The supersymmetry constraints on moduli are

$$T^1 = T^2 \quad \text{and} \quad U^1 = U^2, \quad (4.64)$$

from $\partial W / \partial U^A = 0$ and $\partial W / \partial T^a = 0$, respectively. The Minkowski condition $W = 0$ is then satisfied automatically.

Let us interpret these constraints in the orbifold limit of $T^6 / (\mathbb{Z}_2 \times \mathbb{Z}_2)$, setting all axionic moduli B^a and $C_{(3)}^A$ equal to zero, for simplicity. The constraints become

$$r_1 r_2 = r_3 r_4 \quad \text{and} \quad r_1 / r_2 = r_3 / r_4, \quad (4.65)$$

where the r_i are the radii of $T_{(3)}^4$. From the discussion at the end of appendix A, the $K3$ moduli G_{ij} are related to r_4 and the metric g_{ij} on $T^3(x^1, x^2, x^3)$ via

$$G_{ij} = g_{ij}(r_4 / \sqrt{g}) = g_{ij} / (r_1 r_3). \quad (4.66)$$

Thus,

$$(R_1)^2 = r_1/r_3 = 1 \quad \text{and} \quad (R_3)^2 = r_3/r_1 = 1. \quad (4.67)$$

So, in the metric G_{ij} , the radii R_1 and R_3 are stabilized to unity, by the nontrivial $\text{SO}(2,2)$ monodromy over $S^1(x^5)$.

This is identical to the stabilization of T^2 radii to unity in the nongeometric ‘‘T-fold’’²⁰ fibration of T^2 over S^1 with monodromy

$$\rho, \tau \rightarrow -1/\rho, -1/\tau, \quad (4.68)$$

where ρ and τ are the Kähler and complex structure moduli of the T^2 . In fact, this T-fold background and our twisted $(K3 \times T^2)/\mathbb{Z}_2$ type IIA orientifold are dual — up to the additional circles, orientifold and $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold included in the latter [46].

4.5 Supersymmetric Anti de Sitter vacua

Now let us relax the condition that $W = 0$ and consider generic supersymmetric vacua. The connection K_{Φ^P} that appears in the Kähler covariant derivatives is found to be

$$\begin{aligned} K_{,a} &= -\frac{2}{\sqrt{8V_X}} \int \omega_a \wedge J \wedge J, \\ K_{,A} &= -2e^{K_2/2} \int \alpha_A \wedge \text{Im}(C\hat{\Omega}) = -2e^{K_2/2} \int \alpha_A \wedge \text{Im}(e^{-\phi}\Omega). \end{aligned} \quad (4.69)$$

If we define $\mu = e^{K/2}W/\sqrt{8V_X}$, so that the cosmological constant is $-|\mu|^2$, then the supersymmetry conditions become

$$\begin{aligned} 0 &= D_a W = i \int \omega_a \wedge (d\Omega_c + e^{J_c} \wedge \bar{F}_{\text{RR}} + i\mu e^{-2\phi} J \wedge J), \\ 0 &= D_A W = -i \int \alpha_A \wedge (\bar{H} + dJ_c + 2i\mu \text{Im}(e^{-\phi}\Omega)). \end{aligned} \quad (4.70)$$

Thus,

$$\begin{aligned} d\Omega_c + e^{J_c} \wedge \bar{F}_{\text{RR}} + i\mu e^{-2\phi} J \wedge J &= 0, \\ \bar{H} + dJ_c + 2i\mu \text{Im}(e^{-\phi}\Omega) &= 0, \end{aligned} \quad (4.71)$$

with imaginary parts

$$\begin{aligned} d(e^{-\phi} \text{Re} \Omega) + J \wedge \tilde{F}_{(2)} + \text{Re}(\mu) e^{-2\phi} J \wedge J &= 0, \\ dJ + 2 \text{Re}(\mu) e^{-\phi} \text{Im} \Omega &= 0, \end{aligned} \quad (4.72)$$

²⁰The D-brane spectrum in this 7D T-fold background was recently studied in ref. [35]. For T-folds and other duality twists of elliptic monodromy $(\Gamma_n)^k = 1$, Hellerman and Walcher have given a complete 1-loop characterization of the full string theory background as a generalization of the standard orbifold construction. As a special case, the particular T-fold just described has been shown to yield a modular invariant partition function in type II string theory and to preserve 16 supercharges [27].

and real parts

$$\begin{aligned} \tilde{F}_{(4)} - \frac{1}{2} J \wedge J F_{(0)} - \text{Im}(\mu) e^{-2\phi} J \wedge J &= 0, \\ H - 2 \text{Im}(\mu) e^{-\phi} \text{Im} \Omega &= 0. \end{aligned} \tag{4.73}$$

We will not present examples of AdS vacua here, but note that the general equations (4.71) were studied in ref. [21]. Examples in closely related contexts appear in refs. [4, 36].

5. Geometric twists for Calabi-Yau manifolds of Voisin-Borcea type

The class of Voisin-Borcea Calabi-Yau manifolds of the form $(K3 \times T^2)/\mathbb{Z}_2$ [44] contains many more manifolds than $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. This class is discussed in more detail in appendix C. For our purposes here, we note that each Voisin-Borcea Calabi-Yau manifold is defined, in part, by \mathbb{Z}_2 involution of the second cohomology lattice

$$H^2(K3, \mathbb{Z}) = (-E_8 \times E_8) \times U_{1,1}^3. \tag{5.1}$$

Here, $U_{1,1}$ is a two dimensional lattice of signature (1, 1). For the $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ case above, we have

$$\begin{aligned} H_+^2 &= (-E_8 \times E_8) \times U_{1,1} \quad (\text{from } \chi_I; \chi_3, \chi^3), \\ H_-^2 &= (U_{1,1})^2 \quad (\text{from } \chi_1, \chi^1; \chi_2, \chi^2). \end{aligned} \tag{5.2}$$

However, the twists described above easily extend to other involutions, which lead to different H_+^2 and H_-^2 . Following standard notation, we let r denote the rank of the even part of the cohomology lattice H_+^2 .

Let us assume for simplicity that the $U_{1,1}^3$ factor in H^2 still decomposes into $U_{1,1}$ in H_+^2 and $U_{1,1}^2$ in H_-^2 . Then, the parity of the χ_i and χ^i is unchanged, but the χ_I of the $E_8 \times E_8$ factor in general decompose into $r - 2$ elements χ_{I^+} of H_+^2 and $18 - r$ elements χ_{I^-} of H_-^2 . (For $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, we have $r = 18$ and all χ_I are in H_+^2 .) So, the result (3.33) generalizes to

$$\begin{aligned} \text{Moduli:} & \quad \delta A^{I^+3}, \\ \text{Twists:} & \quad A^{I^+i} = m_n^{I^+i} x^n, \quad i = 1, 2, \quad n = 5, 6, \quad m_n^{I^+i} \in 4\mathbb{Z}, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} \text{Moduli:} & \quad \delta A^{I^-1}, \delta A^{I^-2}, \\ \text{Twists:} & \quad A^{I^-3} = m_n^{I^-3} x^n, \quad n = 5, 6, \quad m_n^{I^-i} \in 4\mathbb{Z}. \end{aligned} \tag{5.4}$$

Here we have assumed the parabolic case, in the terminology of section 3.3. The twist data $m_n^{I^+i}$ and $m_n^{I^-3}$ again partially characterizes the fibration of $K3$ over T^2 . Eqs. (5.3) and (5.4) indicate which components are compatible with the \mathbb{Z}_2 involution. The subset of γ_{nj}^i and β_n^{ij} preserved by the involution is the same as that for $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$: we require one of i, j to be 3 and the other to be 1 or 2. To ensure that the \mathbb{Z}_2 quotient of the $K3$

fibration over T^2 remains well defined as a fibration over $\mathbb{P}^1 = T^2/\mathbb{Z}_2$, we again require that $\gamma_{nj}^i, \beta_n^{ij}, \frac{1}{2}m_n^{I+i}$ and $\frac{1}{2}m_n^{I-3}$ be even integers.

For the basis of differential forms, we choose a very similar basis to that of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. For the 2-forms ω_a , the two differences are: (1) the $K3$ blow-up modes ω_{3I+} are now labeled by I^+ only, and (2) instead of ω_{1I} and ω_{2I} , we now have an involution-dependent set of 2-forms $\omega_{I'}$, that result from blowing up the singularities of $(K3 \times T^2)/\mathbb{Z}_2$. For the 3-forms α_A and β^A , the analogous differences are: (1) we now have 3-forms from the $K3$ complex structure deformations generated by by the ζ_{I-} , for which choose the symplectic basis

$$\alpha_{I-} = -dx^5 \wedge \zeta_{I-}, \quad \beta^{J-} = dx^6 \wedge \zeta_{I-} \delta^{I-J-}, \quad (5.5)$$

and (2) in addition, we have an involution-dependent set of 3-forms that generate complex structure deformations of the singularities of $(K3 \times T^2)/\mathbb{Z}_2$.

Without any assumption on whether the twists are parabolic, elliptic or otherwise, the result (3.37) generalizes to

$$M_a^A = \left(\begin{array}{cccc|cc} \gamma_{53}^2 & -h_{631} & \gamma_{63}^1 & -h_{532} & 0 & n_{53}^{K-} \\ \beta_5^{32} & -\gamma_{61}^3 & \beta_6^{31} & -\gamma_{52}^3 & 0 & m_5^{K-3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ m_5^{J+2} & n_{61}^{J+} & m_6^{J+1} & n_{52}^{J+} & 0 & f_5^{J+K-} \end{array} \right) \quad (5.6)$$

and

$$N_{aA} = \left(\begin{array}{cccc|cc} h^{632} & \gamma_{53}^1 & -h_{531} & -\gamma_{63}^2 & 0 & -n_{63}^{K-} \\ \gamma_{62}^3 & \beta_5^{31} & \gamma_{51}^3 & -\beta_6^{32} & 0 & m_6^{K-3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ -n_{62}^{J+} & m_5^{J+1} & n_{51}^{J+} & -m_6^{J+2} & 0 & f_6^{J+K-} \end{array} \right) \quad (5.7)$$

Here, the vanishing rows correspond to ω_3 and the 2-forms of type (2) above. The vanishing column corresponds to the 3-forms of type (2) above.

6. Further generalizations

6.1 Geometric twists of other $K3$ fibered Calabi-Yau manifolds

Having discussed the geometric twists of $X = (K3 \times T^2)/\mathbb{Z}_2$ from the point of view of the $K3 \times T^2$ covering space, let us now describe the twists more intrinsically from the point of view of the resulting Calabi-Yau quotient X . As we will see, this leads to a natural generalization beyond the Voisin-Borcea class. For generic points on the T^2 in the covering space, the $K3$ surface at a point (x^5, x^6) on T^2 is identified with another $K3$ surface at the point $(-x^5, -x^6)$ via the \mathbb{Z}_2 involution. Thus, the quotient X is a fibration over $T^2/\mathbb{Z}_2 = \mathbb{P}^1$, with generic fiber $K3$ and with singular fibers at the four fixed points on the base. This statement remains true if we instead begin with a nontrivial fibration of $K3$ over T^2 as in the twisted geometry.

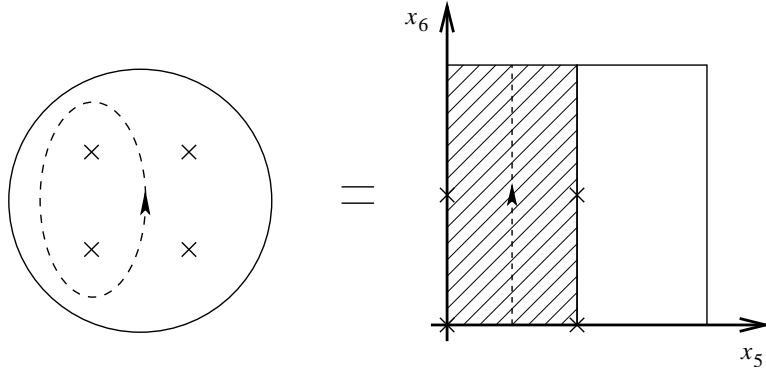


Figure 1: In the identification \mathbb{P}^1 (left) = T^2/\mathbb{Z}_2 (right), the monodromies of the $K3$ moduli about homologically nontrivial circles in T^2 become monodromies about the \mathbb{Z}_2 fixed points in \mathbb{P}^1 .

The twists of eqs. (5.3) and (5.4) can be equivalently expressed as the monodromies of the $K3$ moduli $A_i^{I^+}$ and $A_i^{I^-}$ about the x^5 and x^6 circles in T^2 :

$$\begin{aligned} A^{I^+i} &\rightarrow A^{I^+i} + m_5^{I^+i} & \text{and} & & A^{I^-3} &\rightarrow A^{I^-3} + m_5^{I^-3} & \text{for} & & x^5 &\rightarrow x^5 + 1, \\ A^{I^+i} &\rightarrow A^{I^+i} + m_6^{I^+i} & \text{and} & & A^{I^-3} &\rightarrow A^{I^-3} + m_6^{I^-3} & \text{for} & & x^6 &\rightarrow x^6 + 1. \end{aligned} \quad (6.1)$$

After performing the \mathbb{Z}_2 identification, the x^5 and x^6 circles cease to exist as homology cycles in the \mathbb{P}^1 , but monodromies in eq. (6.1) survive as monodromies about the four \mathbb{Z}_2 fixed points.

Let p_1, p_2, p_3 and p_4 denote the four fixed points with T^2 coordinates $(0,0)$, $(\frac{1}{2},0)$, $(\frac{1}{2},\frac{1}{2})$ and $(0,\frac{1}{2})$, respectively (i.e., counterclockwise starting in the lower left in figure 1). Then, the monodromies about these fixed points are

$$\begin{aligned} M_1 : & \quad A^{I^+i} \rightarrow A^{I^+i} + \frac{1}{2}m_5^{I^+i} + \frac{1}{2}m_6^{I^+i} & \text{and} & & A^{I^-3} &\rightarrow A^{I^-3} + \frac{1}{2}m_5^{I^-3} + \frac{1}{2}m_6^{I^-3}, \\ M_2 : & \quad A^{I^+i} \rightarrow A^{I^+i} + \frac{1}{2}m_5^{I^+i} - \frac{1}{2}m_6^{I^+i} & \text{and} & & A^{I^-3} &\rightarrow A^{I^-3} + \frac{1}{2}m_5^{I^-3} - \frac{1}{2}m_6^{I^-3}, \\ M_3 : & \quad A^{I^+i} \rightarrow A^{I^+i} - \frac{1}{2}m_5^{I^+i} - \frac{1}{2}m_6^{I^+i} & \text{and} & & A^{I^-3} &\rightarrow A^{I^-3} - \frac{1}{2}m_5^{I^-3} - \frac{1}{2}m_6^{I^-3}, \\ M_4 : & \quad A^{I^+i} \rightarrow A^{I^+i} - \frac{1}{2}m_5^{I^+i} + \frac{1}{2}m_6^{I^+i} & \text{and} & & A^{I^-3} &\rightarrow A^{I^-3} - \frac{1}{2}m_5^{I^-3} + \frac{1}{2}m_6^{I^-3}. \end{aligned} \quad (6.2)$$

As a check, note that M_1M_4 gives the monodromy

$$M_1M_4 : \quad A^{I^+i} \rightarrow A^{I^+i} + m_6^{I^+i} \quad \text{and} \quad A^{I^-3} \rightarrow A^{I^-3} + m_6^{I^-3},$$

which is indeed the correct monodromy about the x^6 circle of figure 6. Likewise $(M_2M_3)^{-1}$ gives the monodromy about the x^6 circle, while M_1M_2 and $(M_3M_4)^{-1}$ give the correct monodromy about the x^5 circle. More generally, given monodromies $\Gamma_5(1) = (\Gamma_5(\frac{1}{2}))^2$ and $\Gamma_6(1) = (\Gamma_5(\frac{1}{2}))^2$ about the x^5 and x^6 circles of T^2 , the monodromies about the four fixed

points are

$$\begin{aligned} \Gamma_{p_1} &= \Gamma_5\left(\frac{1}{2}\right)\Gamma_6\left(\frac{1}{2}\right), & \Gamma_{p_2} &= \Gamma_5\left(\frac{1}{2}\right)\Gamma_6\left(-\frac{1}{2}\right), \\ \Gamma_{p_3} &= \Gamma_5\left(-\frac{1}{2}\right)\Gamma_6\left(-\frac{1}{2}\right), & \Gamma_{p_4} &= \Gamma_5\left(-\frac{1}{2}\right)\Gamma_6\left(\frac{1}{2}\right). \end{aligned} \tag{6.3}$$

There is a natural generalization to other $K3$ fibered Calabi-Yau manifolds. For any such fibration, there is a set of points p_i on the base \mathbb{P}^1 over which the $K3$ fiber degenerates, and an associated set of $\Gamma_{3,19}$ monodromies M_i that give the automorphisms of the homology lattice of the $K3$ fibers that result from circling these points. We can alter these monodromies. Provided that the total monodromy about all of the p_i is trivial, we obtain another well defined manifold, which in general is non Calabi-Yau.

A very similar idea was discussed by in ref. [41], in the case of mirror of the quintic hypersurface in \mathbb{P}^4 , viewed as a T^3 fibration over S^3 . Here, the twists instead represent modified monodromies about singular loci on the S^3 . This idea extends in principle to other Calabi-Yau manifolds, provided that the Strominger-Yau-Zaslow T^3 fibration is known. However, the degenerate locus on the base in this case consists of a one dimensional web in S^3 as opposed to a collection of points in \mathbb{P}^1 . (For the quintic, this web is known and has been described by M. Gross [24], as summarized in [41].) Therefore, in practice, explicit realizations seem easier in the $K3$ fibered context described here. Of course, for Calabi-Yau manifolds that are both $K3$ and T^3 fibered, the two constructions should agree. Note, however, that $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ does not fall into this category since the mirror is not purely geometric, but is instead an orbifold with discrete torsion.

6.2 Nongeometric twists

Let us return to the \mathbb{Z}_2 covering space description of the twists for the Voisin-Borcea class. The geometric twists are defined by a nontrivial fibration of $K3$ over T^2 . As discussed in section 3.3, compactification of string theory or supergravity on the twisted geometry can be viewed as a Scherk-Schwarz compactification — a two step compactification, in which the moduli from the first step of the compactification (on $K3$) are given nontrivial dependence on the coordinates of the second second step of the compactification (on T^2).²¹ Compactification of type IIA string theory or supergravity on $K3$ includes more than just metric moduli. The massless spectrum consists of the $\mathcal{N} = (1, 1)$ 6D gravity multiplet and 20 vector multiplets. The vector multiplets each contain 4 scalars, so there are a total of 80 scalars in matter multiplets. Of these, 58 are accounted for by the $K3$ metric moduli

²¹Here, we use the term Scherk-Schwarz *compactification* as opposed to Scherk-Schwarz *reduction* to distinguish the full twisted compactification from its truncation to a particular subsector. See ref. [28] and the Introduction of ref. [30] for a discussion of this point. In their terminology, a two step compactification, with a clear fiber theory and base, would be termed a duality twist. The twisted tori discussed at the beginning of section 3.2 are not necessarily of this type, in that they include, for example, the $SU(2)$ group manifold S^3 for $\gamma_{jk}^i = \epsilon^i_{jk}$. Although S^3 can be viewed as a Hopf fibration of S^1 over S^2 , when viewed as a twisted T^3 it has no clear fiber and base: the γ_{jk}^i parametrize twists of all three S^1 factors over all three complementary T^2 factors.

space (3.10) and 22 by the moduli space $H^2(K3, \mathbb{U}(1)) \sim H^2(K3, \mathbb{R})/H^2(K3, \mathbb{Z})$ of the NS B -field. The one remaining scalar is the dilaton, in the gravity multiplet.

After accounting for all discrete identifications, the complete moduli space for type IIA compactification on $K3$ is²²

$$\mathcal{M}_{\text{IIA}} = \mathbb{R}_{>0} \times (\text{SO}(4) \times \text{SO}(20)) \backslash \text{SO}(4, 20) / \Gamma_{4,20}. \quad (6.4)$$

In the second step of the Scherk-Schwarz compactification, compactification on T^2 , we need not restrict ourselves to the geometric duality group of the fiber theory, $\Gamma_{3,19}$, but can instead allow arbitrary commuting monodromies $\Gamma_5(1), \Gamma_6(1) \in \Gamma_{4,20}$ under $x^5 \rightarrow x^5 + 1$ and $x^6 \rightarrow x^6 + 1$. Since these monodromies in general mix the metric and B -field moduli, the compactifications are in general nongeometric [25]. This construction is similar to the T-fold construction of ref. [29]. In fact there should be a similar, “partially doubled geometry” description of these compactifications analogous to the doubled torus of [29]. To linearize the action of the duality group $\Gamma_{4,20}$, only the part inherited from T^4 , contributing duality group $\Gamma_{4,4}$, actually needs to be doubled.

A still more general nongeometric construction, would be to allow $\Gamma_{4,20}$ monodromies not only over the torus $T^2(x^5, x^6)$, but simultaneously over the T-dual torus $T^2(\tilde{x}^5, \tilde{x}^6)$. Constructions of this type have been discussed in ref. [13]. Again, both here and in the previous paragraph, there is a natural generalization to compactifications on arbitrary $K3$ fibered Calabi-Yau manifolds.

6.3 Comments on effective field theory

A critical property for all of the twisted compactifications discussed here is that there is a natural split into fiber and base. For the $\mathcal{N} = 2$ pre orientifold theory, the $K3$ fibration over T^2 gives rise to precisely the right twist data to parametrize couplings of vector multiplets to hypermultiplets in an $\mathcal{N} = 2$ gauged supergravity theory. In the $(K3 \times T^2)/\mathbb{Z}_2$ orbifold untwisted sector, the vector multiplet moduli space is classically an $\text{SL}(2) \times \text{SO}(2, r)$ coset and the hypermultiplet moduli space is a $\text{SO}(4, 22 - r)$ coset. (Here, r is the Voisin-Borcea parameter introduced in section 5 and appendix B, with $r = 18$ for $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.) The couplings are the $(\text{SL}(2) \times \text{SO}(2, r), \text{SO}(4, 22 - r))$ bifundamentals whose moduli are projected out by the orbifold \mathbb{Z}_2 . For T^2 fibrations over $K3$ there is likewise a natural split.

From the point of view of the parent theory on $T^2 \times K3$, there is no such natural split in the $\mathcal{N} = 4$ low energy effective theory, despite the fact that there is one in the target space geometry. The moduli space is an $\text{SL}(2) \times \text{SO}(6, 6 + 16)$ coset, as is clear from the dual heterotic description on T^6 . Consequently, the gauged $\mathcal{N} = 4$ supergravity data for the $K3$ fibration over T^2 represents, from the point of view of the effective field theory, an arbitrary choice of decomposition:

$$\text{SL}(2) \times \text{SO}(6, 6 + 16) \quad \rightarrow \quad (\text{SL}(2) \times \text{SO}(2, r)) \times \text{SO}(4, 22 - r). \quad (6.5)$$

Finally, note that the construction by Tomasiello [41] employs a different fiber/base split of T^3 over S^3 . Thus, although there is an overlapping class of twisted geometries that

²²In fact, there is a beautiful interpretation of this moduli space as the space of positive signature 4-plane in the total cohomology space $H^*(K3, \mathbb{R}) = \mathbb{R}^{4,20}$. See, for example, refs. [43, 2].

are simultaneously $K3$ fibrations over \mathbb{P}^1 and T^3 fibrations over S^3 , both constructions should contain twists that the other does not, and therefore give somewhat different possible couplings in the low energy effective field theory.

7. Conclusions

We have described how discrete geometric twists can be included in the set of defining data for string compactifications based on Calabi-Yau manifolds of Voisin-Borcea type $(K3 \times T^2)/\mathbb{Z}_2$. The twist data define a nontrivial fibration of $K3$ over T^2 compatible with the \mathbb{Z}_2 involution, and a corresponding discrete deformation of the closure and exactness relations of the Calabi-Yau cohomology ring. The data can be parametrized either in terms of monodromies Γ_n in the automorphism group $\Gamma_{3,19}$ of the $K3$ cohomology lattice (under $x^n \rightarrow x^n + 1$ on the T^2), or their logarithms, which appear in the modified closure relations. The quantization conditions on the latter depend on the conjugacy class of Γ_n , as we have illustrated through two concrete cases. We have termed these cases parabolic and elliptic by analogy to the conjugacy classes of $SL(2, \mathbb{R})$.

For the particular Voisin-Borcea manifold $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, which features prominently in model building, we have studied the type IIA orientifold of the twisted background in detail. Supersymmetric Minkowski vacua are of the holomorphic monopole form [34, 20] and lift to M theory compactified on manifolds of G_2 holonomy. We have presented four examples of such vacua, three of parabolic type with nonzero Ramond-Ramond flux and one of elliptic type with no flux. A feature that we observed is that even when the discrete data involves the orbifold untwisted sector only, the blow-up moduli can nevertheless be stabilized due to the interplay between orbifold untwisted and twisted sectors in the intersection form of the Calabi-Yau. The first three examples share the property that all of the blow-up Kähler moduli of $(K3 \times T^2)/\mathbb{Z}_2$ are classically stabilized to zero. By analogy to conformal field theory results in the context of the heterotic string on T^6/\mathbb{Z}_3 [10], we suspect that this result persists in the presence of worldsheet instanton corrections. However, this clearly needs to be explored further. We are currently investigating the requisite CFT expansion about the orbifold limit, for small blow-up Kähler moduli.

There are several other possible directions for future work. First, the geometric twists constructed here include only one third of the blow-up modes of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$: those that resolve $T^4_{(3)}/\mathbb{Z}_2$ to $K3_{(3)}$, to give $(K3 \times T^2)/\mathbb{Z}_2$. Our construction can never twist the exceptional cohomology of $(K3 \times T^2)/\mathbb{Z}_2$. As discussed in section 6, the monodromies introduced here are monodromies *around* the \mathbb{Z}_2 fixed points on the base $\mathbb{P}^1 = T^2/\mathbb{Z}_2$, while the exceptional cohomology is associated with singular fibers localized *at* the fixed points on the base. It would be interesting to find a different way to twist the geometry that can include the $(K3 \times T^2)/\mathbb{Z}_2$ blow-up modes and exceptional complex structure deformations.

While the main goal of our work was to construct, for the Voisin-Borcea class, the geometric twists described in section 2, we were soon confronted with the larger challenge of understanding the class of type IIA Calabi-Yau orientifolds with Ramond-Ramond flux and geometric twists. There is currently a gap in the literature in terms of even qualitatively

understanding vacua of this type. Nevertheless, this class is the natural geometric analog in type IIA of Calabi-Yau orientifolds in type IIB with Neveu-Schwarz and Ramond-Ramond fluxes. These IIA and IIB classes become mirror to one another only after nongeometric twists are included as well, however, the simplest subclasses that one could hope to understand are the geometric subclasses for either IIA and IIB. Much work has been done for IIB with fluxes but very little for IIA with flux and twists (see for example the reviews [17, 6]). This is a clear avenue for future work.

Finally, a great deal of our understanding of the space of string theory vacua has come from duality, with type IIA/heterotic duality featuring prominently in our understanding of type IIA vacua based on $K3$ fibrations. Since the central aim of our work was to understand the discrete geometric data that can be incorporated into vacua based on $K3$ fibrations, it is natural to ask what the dual heterotic description of this data is. As discussed in section 3.3, the geometric twists of $(K3 \times T^2)/\mathbb{Z}_2$ analyzed here represent Scherk-Schwarz twists of $K3$ moduli upon further compactification on the T^2 . For the twists surviving the \mathbb{Z}_2 projection, the corresponding continuous moduli are projected out. In the heterotic dual, one analog of this is heterotic flux for which the orbifold projects out the corresponding zero modes of the gauge fields. The complete duality map and dual heterotic description is currently under investigation [46].

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A. HyperKähler structure on T^4

A choice of hyperKähler structure on T^4 is analogous to a choice of complex structure on T^2 . Let us first review the latter in a way that makes the generalization natural, and then go on to discuss hyperKähler structure on T^4 . This review is taken more or less directly from ref. [50], currently in preparation by one of the authors.

On T^2 , we can express the metric as

$$ds_{T^2}^2 = e^1 \otimes e^1 + e^2 \otimes e^2, \quad (\text{A.1})$$

where $e^m = e^m_n dx^n$ in terms of the vielbein e^m_n . The complex structure is defined by a

tensor J_i^j which we view as a map²³ $J: T^* \rightarrow T^*$, such that

$$J: \quad e^2 \rightarrow e^1, \quad e^1 \rightarrow -e^2. \quad (\text{A.2})$$

Lowering the upper index of J_m^n gives the Kähler form on T^2 . By $\text{SL}(2, \mathbb{Z})$ change of lattice basis for the lattice Λ that enters into $T^2 = \mathbb{R}^2/\Lambda$, we can always write

$$\begin{aligned} e^1 &= R^1(dx^1 + a^1_2 dx^2), \\ e^2 &= R^2 dx^2, \quad \text{where } x^n \cong x^n + 1. \end{aligned} \quad (\text{A.3})$$

The holomorphic 1-form is

$$e^z = e^1 + ie^2 = R^1(dx^1 + \tau_1 dx^2), \quad (\text{A.4})$$

where $\tau_1 = a^1_2 + i\frac{R^2}{R^1}$ is the complex structure modulus.

Likewise, we can express the metric on T^4 as

$$ds_{T^4}^2 = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4, \quad (\text{A.5})$$

where, again, $e^m = e^m_n dx^n$ in terms of the vielbein e^m_n . The hyperKähler structure is defined by a triple of tensors $J_{(i)m}^n$, $i = 1, 2, 3$, which we view as maps $J_{(i)}: T^* \rightarrow T^*$, such that

$$\begin{aligned} J_{(1)}: \quad & e^4 \rightarrow e^1, \quad e^3 \rightarrow e^2, \quad e^1 \rightarrow -e^4, \quad e^2 \rightarrow -e^3, \\ J_{(2)}: \quad & e^4 \rightarrow e^2, \quad e^1 \rightarrow e^3, \quad e^2 \rightarrow -e^4, \quad e^3 \rightarrow -e^1, \\ J_{(3)}: \quad & e^4 \rightarrow e^3, \quad e^2 \rightarrow e^1, \quad e^3 \rightarrow -e^4, \quad e^1 \rightarrow -e^2. \end{aligned} \quad (\text{A.6})$$

The $J_{(i)m}^n$ satisfy

$$J_{(1)}J_{(2)} = -J_{(2)}J_{(1)} = -J_{(3)}, \quad (J_{(1)})^2 = -1,$$

plus cyclic permutations. Lowering the upper index on $J_{(i)m}^n$ gives a triple of Kähler forms $J_{(i)mn}$. The quaternionic 1-form is

$$e^a = e^4 - \mathbf{i}e^1 - \mathbf{j}e^2 - \mathbf{k}e^3, \quad (\text{A.7})$$

where the quaternions $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy the same algebra as $-J_{(i)}$:²⁴

$$\mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}, \quad \text{and} \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1. \quad (\text{A.8})$$

²³The usual convention in the math literature is the transpose of this: J has index structure J^m_n , so that J acts from the left on the tangent space, and J^T acts from the left on the cotangent space. However, if we require that (i) J with holomorphic (antiholomorphic) indices be $+i$ ($-i$), as is customary in both the math and physics conventions, and (ii) the Kähler form be obtained by lowering one index of the tensor J , with no sign change, then we are uniquely led to the conventions used in this paper.

²⁴Here, $-J_{(i)}$ rather than $J_{(i)}$ satisfies the quaternion algebra for the reason discussed in the previous footnote. Note that the tangent space map $J_{(i)}^T$ satisfies the quaternion algebra with no minus sign.

A choice of complex structure on T^4 is then a choice of i on the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ unit sphere. By a $\text{SL}(4, \mathbb{Z})$ change of lattice basis for the T^4 , we can write, in addition to eq. (A.3),

$$\begin{aligned} e^3 &= R^3(dx^3 + a^3_1 dx^1 + a^3_2 dx^2), \\ e^4 &= R^4(dx^4 + a^4_1 dx^1 + a^4_2 dx^2 + a^4_3 dx^3), \quad \text{where } x^n \cong x^n + 1. \end{aligned} \quad (\text{A.9})$$

So, for example, if we choose complex structure $i = \mathbf{k}$, then the complex pairing that follows from $J = J_{(3)}$ is

$$\begin{aligned} e^{z^1} &= R_1 e^1 + i R_2 e^2 = R^1(dx^1 + \tau_1 dx^2), \\ e^{z^2} &= R_4 e^4 - i R_3 e^3 = R^4(dx^4 + \tau_2^{-1} dx^3 + \dots), \end{aligned} \quad (\text{A.10})$$

where $\tau_2^{-1} = a^4_3 - i \frac{R_3}{R_4}$ and the “...” is a 1-form on $T^2(x^1, x^2)$, which can be interpreted as the connection for a trivial fibration of $T^2(x^3, x^4)$ over $T^2(x^1, x^2)$. The holomorphic $(2, 0)$ form in this case is

$$\Omega_{(2,0)} = e^{z^1} \wedge e^{z^2} = J_{(1)} + i J_{(2)}. \quad (\text{A.11})$$

If we write the metric on T^4 as

$$ds_{T^4}^2 = R_4^2(dx^4 + a_i^4 dx^i)^2 + g_{ij} dx^i dx^j, \quad i = 1, 2, 3, \quad (\text{A.12})$$

then the choice of hyperKähler structure is the choice of a^4_i together with the dimensionless metric $G_{ij} = (R_4/\sqrt{g})g_{ij}$. Let us define $B^{ij} = -a^4_k \epsilon^{kij}$, where $\epsilon^{123} = 1$. Then, this choice parametrizes the $(\text{SO}(3) \times \text{SO}(3)) \backslash \text{SO}(3, 3)/\Gamma_{3,3}$ truncation of the coset (3.13), with vielbein

$$V = \begin{pmatrix} E & -EB \\ 0 & E^{-1T} \end{pmatrix}, \quad (\text{A.13})$$

where E^λ_i is the vielbein for the metric G_{ij} . The coset can also be interpreted as the choice of positive signature 3-plane spanned by $J_{(1)}, J_{(2)}, J_{(3)}$ in $H^2(T^4, \mathbb{R}) = \mathbb{R}^{3,3}$.

B. The homology lattices of T^4/\mathbb{Z}_2 and $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$

For completeness, we review the integer homology lattices of T^4/\mathbb{Z}_2 and $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. This review is based primarily on refs. [15] and [43].

B.1 The lattice of T^4/\mathbb{Z}_2

Let us view T^4 as $T_{(1)}^2(x^1, x^2) \times T_{(2)}^2(x^3, x^4)$ with complex pairing $dz_1 = dx^1 + \tau_1 dx^2$ and $dz_2 = dx^3 + \tau_2 dx^4$. Now consider T^4/\mathbb{Z}_2 . There are $2^4 = 16$ points of local geometry $\mathbb{C}^2/\mathbb{Z}_2$ (16 A_1 singularities), located at the fixed points where each of the four coordinates is equal to 0 or $1/2$. There are also $4 + 4 = 8$ fixed lines \mathbb{P}^1 with a simple description in this complex structure: let D_{1s} , $s = 1, 2, 3, 4$ label the divisors $\mathbb{P}^1 = T_{(2)}^2/\mathbb{Z}_2$ located at each of the four fixed points in (x^1, x^2) and D_{2t} denote the divisors $\mathbb{P}^1 = T_{(1)}^2/\mathbb{Z}_2$ located at the four fixed point in (x^3, x^4) . The intersections of these \mathbb{P}^1 s in the singular geometry is illustrated schematically in figure 2 (a).

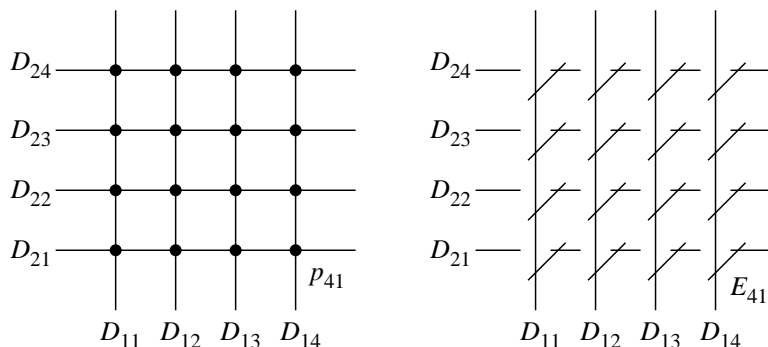


Figure 2: (a) In the singular T^4/\mathbb{Z}_2 (left), each of the sixteen A_1 singularities is the “half point” of intersection, p_{st} , of two fixed \mathbb{P}^1 s, D_{1s} and D_{2t} . (b) In the resolved $K3$ (right), each p_{st} is blown up to an exceptional divisor E_{st} . After resolution, D_{1s} and D_{2t} no longer intersect, but each intersects E_{st} in a point. In the figures above, only p_{41} and its blow up E_{41} are labeled explicitly.



Figure 3: (a) The fan for the local model $\mathbb{C}^2/\mathbb{Z}_2$ at the singular point p_{st} in T^4/\mathbb{Z}_2 (left), and (b) the fan for the resolution (right), with the point p_{st} blown up to the exceptional divisor E_{st} .

The homology classes of the $D_{\alpha i}$ in the singular geometry are

$$D_{1s} = \frac{1}{2}f_1, \quad D_{2t} = \frac{1}{2}f_2, \quad \text{independent of } s, \quad (\text{B.1})$$

where f_α is the class of $T^2_{(\alpha)}$. Let us focus on the singularity at the “half point”²⁵ $p_{st} = D_{1s} \cap D_{2t}$, and consider the local model $\mathbb{C}^2/\mathbb{Z}_2$ at this point.

Figure 3 (a) gives the fan for the toric description of $\mathbb{C}^2/\mathbb{Z}_2$. There is a single two dimensional fan of volume 2 generated by the lattice vectors $D_{1s} = (0, 1)$ and $D_{2t} = (2, 1)$, each of which represents a divisor of T^4/\mathbb{Z}_2 . If we take p_{st} to be the origin of $\mathbb{C}^2/\mathbb{Z}_2$, then these divisors are $D_{1s} = \{z_1 = 0\}$ and $D_{2t} = \{z_2 = 0\}$. In the toric description, to resolve the singularity, we subdivide the original singular cone into two cones of volume 1 by introducing a new divisor E_{st} . E_{st} is the exceptional divisor obtained by blowing up the origin of $\mathbb{C}^2/\mathbb{Z}_2$.

Let us make this more explicit. To each of the lattice components r , we associate a monomial $U_r = \prod_i z_i^{(D_i)_r}$, where $(D_i)_r$ is the r th component of the lattice vector D_i in the fan. The toric variety is then given by the set of all (z_1, z_2) not in the excluded set F modulo rescalings that leave the U_r invariant. The excluded set F consists of all points that have simultaneous zeros of coordinates whose corresponding D_i do not lie in the same cone. For the unresolved fan of figure 3 (a), there is just a single two dimensional cone, so $F = \emptyset$. The only rescaling that leaves U_1, U_2 invariant is $\mathbb{Z}_2: (z_1, z_2) \rightarrow (-z_1, -z_2)$. So, the toric variety is indeed $\{(z_1, z_2)\}/\mathbb{Z}_2 = \mathbb{C}^2/\mathbb{Z}_2$.

²⁵This “half point” is the interpretation of $\int_{K3} (dx^1 \wedge dx^2) \wedge (dx^3 \wedge dx^4) = \frac{1}{2} \int_{T^4} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 = 1/2$.

For the resolved fan, we include the lattice vector $E_{st} = (1, 1)$ as well, as shown in figure 3 (b). In this case, $U_1 = z_2^2 w$ and $U_2 = z_1 z_2 w$, where w is the new coordinate associated to E_{st} . The excluded set is $F = \{z_1 = z_2 = 0\}$. The rescaling symmetry of U_1, U_2 is \mathbb{C}^* : $(z_1, z_2, w) \rightarrow (\lambda z_1, \lambda z_2, \lambda^{-2} w)$. Away from $w = 0$, this gives $(z_1, z_2, 1)/\mathbb{Z}_2 = \mathbb{C}^2/\mathbb{Z}_2$ with the z -origin deleted. At $w = 0$, we obtain the exceptional \mathbb{P}^1 , $E_{st} = \{(z_1, z_2, 0) \setminus (0, 0, 0)\}/\mathbb{C}^*$.

Divisors can always be represented in patches as the vanishing loci of local meromorphic functions. However, divisors that globally have such a representation are homologically trivial and have trivial intersection with other divisors. (See, for example, ref. [47].)

In our toric model for the resolution of $\mathbb{C}^2/\mathbb{Z}_2$, a basis of such global meromorphic functions is U_1, U_2 . The corresponding homologically trivial divisors are $2D_{1s} + E_{st}$ (from $U_2^2/U_1 = 0$) and $2D_{2t} + E_{st}$ (from $U_1 = 0$). In the compact $K3$, (as explained for $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ in ref. [15]), these relations become

$$f_1 = 2D_{1s} + \sum_{t=1}^4 E_{st} \quad \text{independent of } s, \quad f_2 = 2D_{2t} + \sum_{s=1}^4 E_{st} \quad \text{independent of } t, \quad (\text{B.2})$$

where the divisors f_1 and f_2 are not homologically trivial, but instead correspond to “sliding divisors” that can be moved away from the (resolved) singularities. They have trivial intersection with the exceptional divisors E_{st} and represent the tori $f_1 = \{z_1 = c_1\} \cup \{z_1 = -c_1\}$ and $f_2 = \{z_2 = c_2\} \cup \{z_2 = -c_2\}$ on the T^4 covering space, where c_1, c_2 are non fixed points. The corresponding Poincaré dual cohomology classes are $2dx^1 \wedge dx^2$ and $2dx^3 \wedge dx^4$, respectively.

The cycles in $K3$ described so far are those that are particularly simple in the complex structure $J_{(3)}$. In the same way, in the complex structure $J_{(1)}$ we obtain homology classes f_3 and f_4 from elliptic curves $T_{(3)}^2$ and $T_{(4)}^2$ located at non fixed points in (x^1, x^4) and (x^2, x^3) , respectively. In the complex structure $J_{(2)}$ we obtain homology classes f_5 and f_6 from elliptic curves $T_{(5)}^2$ and $T_{(6)}^2$ located at non fixed points in (x^2, x^4) and (x^3, x^1) . Likewise, we obtain divisors D_{3s}, D_{4t} and D_{5s}, D_{6t} by setting the corresponding pairs of coordinates equal to their \mathbb{Z}_2 fixed values before the resolution. The homology lattice of $K3$ is the integer span of the overcomplete basis given by the $6f$, $24D$ and $16E$ divisors.

To make the last paragraph more explicit, it is convenient to use the notation of ref. [43]. Let us label the fixed points by twice their T^4 coordinate values (i.e, by ordered quadruples (y^1, y^2, y^3, y^4) , with $y^i = 2x^i \in \mathbb{F}_2 = \{0, 1\}$). Then the fixed points and exceptional divisors are specified by a point y in \mathbb{F}_2^4 . The divisors D_κ are

$$D_\kappa = \frac{1}{2}\kappa - \frac{1}{2} \sum_{y \in P(\kappa)} E_y, \quad (\text{B.3})$$

where κ is the pullback to $K3$ of a fixed T^2 in T^4 meeting the subset $P \subset \mathbb{F}_2^4$ of fixed points. There are $6 \times 4 = 24$ different κ in 6 homology classes f , from the $\binom{4}{2} = 6$ directions spanned by the T^2 in T^4 and the $2^2 = 4$ fixed locations on the transverse T^2 .

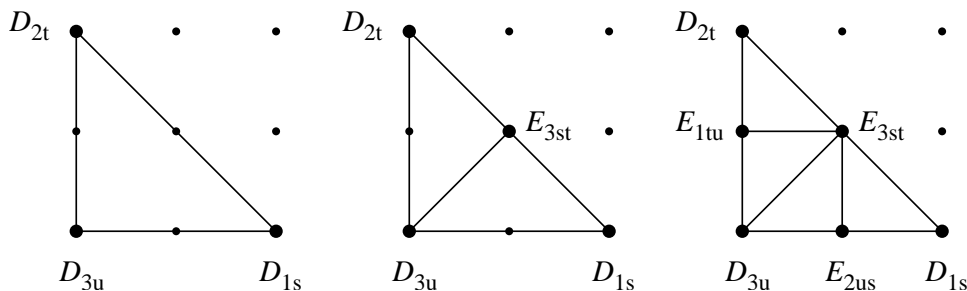


Figure 4: Fans for: (a) the local model $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ at the singular point p_{stu} of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, (b) the partially resolved local model $((\text{Eguchi-Hanson}) \times \mathbb{C})/\mathbb{Z}_2$, and (c) the fully resolved local model.

B.2 The lattice of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$

The homology lattice of the Calabi-Yau orientifold $X = T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is very similar to that just discussed for $K3$, and is in some sense simpler due to the absence of the hyperKähler structure. In this section, we closely follow ref. [15].

The unresolved orbifold $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ has $2^6 = 64$ fixed points of local geometry $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. The fan for the latter is shown in figure 4 (a). There is a single cone of volume 4, with vertices $D_{1s} = (2, 0, 1)$, $D_{2t} = (0, 2, 1)$ and $D_{3u} = (0, 0, 1)$. Here, the second subscript of the divisor $D_{\alpha s}$ take values $s = 1, \dots, 4$ and indicates the location of the divisor among the four fixed points on the transverse space $\mathbb{P}^1 = T^2_{(\alpha)}/\mathbb{Z}_2$. Since we have already reviewed the toric geometry of T^4/\mathbb{Z}_2 above, we will be more telegraphic here. The monomials U_r are $U_1 = z_1^2$, $U_2 = z_2^2$ and $U_3 = z_1 z_2 z_3$, with rescaling symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$, which we take to be generated by σ_3 and σ_1 of eq. (3.1). The excluded set is $F = \emptyset$. So, we indeed obtain $\{(z_1, z_2, z_3)\}/(\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

For the construction described in this paper we resolve X one \mathbb{Z}_2 at a time — first with respect to σ_3 to obtain the Voisin-Borcea orbifold $(K3_{(3)} \times T^2_{(3)})/\sigma_1$, and then with respect to σ_1 . Let us focus on the local model $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ at the “quarter point”²⁶ $p_{stu} = D_{1s} \cap D_{2u} \cap D_{3u}$. The first step, the resolution with respect to σ_3 , introduces the exceptional divisor E_{3st} in the partially resolved fan of figure 4 (b). After this step, there are two cones, each of volume 2. The geometry is $((\text{Eguchi-Hanson}) \times \mathbb{C})/\mathbb{Z}_2$. The second step introduces additional exceptional divisors E_{1tu} and E_{2us} and gives the fully resolved fan shown in figure 4 (c). As explained in ref. [15], this is the *asymmetric* resolution of $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ with distinguished direction $\alpha = 3$. A *symmetric* resolution is also possible, and differs by a flop of the curve $C_{3stu} = D_{3u} \cap E_{3st}$ (the diagonal line in figures 4 (b,c)).

Let w_1, w_2, w_3 denote the toric coordinates associated to $E_{1tu}, E_{2us}, E_{3st}$, respectively. Then, in the resolution of the local model, we obtain the monomials $U_1 = z_1^2 w_2 w_3$, $U_2 = z_2^2 w_3 w_1$ and $U_3 = z_1 z_2 z_3 w_1 w_2 w_3$. The scaling symmetry is

$$\mathbb{C}^{*3}: (z_1, z_2, z_3, w_1, w_2, w_3) \rightarrow \left(\lambda_1 z_1, \lambda_2 z_2, \lambda_3 z_3, \frac{\lambda_1}{\lambda_2 \lambda_3} w_1, \frac{\lambda_2}{\lambda_3 \lambda_1} w_2, \frac{\lambda_3}{\lambda_1 \lambda_2} w_3 \right), \quad (\text{B.4})$$

²⁶This “quarter point” is the interpretation of $\int_X (dx^1 \wedge dx^2) \wedge (dx^3 \wedge dx^4) \wedge (dx^5 \wedge dx^6) = \frac{1}{4} \int_{T^6} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6 = 1/4$.

and the resolved local model is the toric variety $(\mathbb{C}^6 \setminus F)/\mathbb{C}^{*3}$, where F is the excluded set.

From fact that U_1, U_2, U_3 are meromorphic functions in the local model, we obtain the homology relations $2D_{1s} + E_{2us} + E_{3st} = 0$ in the local model, together with cyclic permutations in 1, 2, 3. In the compact Calabi-Yau manifold X , these relations become

$$\begin{aligned}
 F_1 &= 2D_{1s} + \sum_{u=1}^4 E_{2us} + \sum_{t=1}^4 E_{3st} \quad \text{independent of } s, \\
 F_2 &= 2D_{2t} + \sum_{s=1}^4 E_{3st} + \sum_{u=1}^4 E_{1tu} \quad \text{independent of } t, \\
 F_3 &= 2D_{3u} + \sum_{t=1}^4 E_{1tu} + \sum_{s=1}^4 E_{2us} \quad \text{independent of } u.
 \end{aligned} \tag{B.5}$$

Here, the F_α are homologically nontrivial ‘‘sliding divisors,’’ which can be moved away from the (resolved) singularities of X , and therefore do not intersect the exceptional cycles. These are the divisors $K3_{(\alpha)}$ already mentioned in section 3.1.

The integer homology lattice $H^2(X, \mathbb{Z})$ is generated by $\{F_\alpha, D_{\alpha s}, E_{\alpha st}\}$. Using the relations (B.5), the subset $\{F_\alpha, E_{\alpha st}\}$ forms a linearly independent basis, though one that requires some coefficients in $\mathbb{Z}/2$ rather than \mathbb{Z} to form a basis for $H^2(X, \mathbb{Z})$. (For example, $D_{1s} = \frac{1}{2}F_1 - \frac{1}{2}\sum_{u=1}^4 E_{2us} - \frac{1}{2}\sum_{t=1}^4 E_{3st}$.) This is the basis employed throughout the paper, with the following modification: for notational simplicity we use the multi-index I instead of st for the exceptional divisors.

The intersection numbers κ_{abc} for a, b, c distinct can be computed using the local model. The remaining intersections, κ_{aab} and κ_{aaa} then follow from the κ_{abc} together with the relations (B.5), as discussed in ref. [15]. Let us simply quote the result here. If we write the Poincaré dual of the Kähler form as

$$J = v^i F_i - v^{3st} E_{3st} - v^{1tu} E_{1tu} - v^{2us} E_{2us}, \tag{B.6}$$

then the volume of X is

$$\begin{aligned}
 V_X &= \frac{1}{6} \kappa_{abc} v^a v^b v^c = 2v^1 v^2 v^3 - v^3 \sum_{st} (v^{3st})^2 - v^1 \sum_{tu} (v^{1tu})^2 - v^2 \sum_{us} (v^{2us})^2 \\
 &\quad - \frac{4}{3} \left(\sum_{tu} (v^{1tu})^3 + \sum_{us} (v^{2us})^3 \right) + \sum_{stu} v^{3st} \left((v^{1tu})^2 + (v^{2us})^2 \right).
 \end{aligned} \tag{B.7}$$

This is twice the volume given in eq. (6.9) of ref. [15]. Note that for $v^{1tu} = v^{2us} = 0$, which is the case for the partial resolution of figure 4(b), we obtain half of the intersection form of $K3_{(3)} \times T_{(3)}^2$ expressed as a function of the $T_{(3)}^2$ Kähler modulus $2v^3$ and $K3_{(3)}$ Kähler moduli v^a , $a \neq 3$. Likewise for $a = F_\alpha$, the κ_{abc} give the correct intersection numbers κ_{bc} on $K3_{(\alpha)}$.

C. Calabi-Yau manifolds of Voisin-Borcea type

Calabi-Yau manifolds of the form $X = (K3 \times T^2)/\mathbb{Z}_2$ comprise the Voisin-Borcea class [44, 48]. They are conventionally characterized by three integers (r, a, δ) , which we now describe.

The \mathbb{Z}_2 acts by inversion ($\mathcal{I}_2 : x^n \rightarrow -x^n$) on the T^2 and holomorphic involution σ on the $K3$ surface. In order that there exist a $(3, 0)$ form on the quotient X , the involution must act as (-1) on $H^{2,0}(K3)$. Nikulin [49] has classified all such involutions in terms of the integers (r, a, δ) .

The integer r was already discussed in section 5. The second cohomology lattice of $K3$ is

$$H^2(K3, \mathbb{Z}) = (-E_8 \times E_8) \times U_{1,1}^3, \tag{C.1}$$

where $U_{1,1}$ is a two dimensional lattice of signature $(1, 1)$. Under the action of σ , $H^2(K3, \mathbb{Z})$ decomposes into even and odd parts $H_+^2(K3, \mathbb{Z})$ and $H_-^2(K3, \mathbb{Z})$. The integer r gives the rank of $H_+^2(K3, \mathbb{Z})$. In terms of r , the orbifold untwisted sector of X contributes

$$h_{\text{ut}}^{1,1} = r + 1 \quad \text{and} \quad h_{\text{ut}}^{2,1} = 21 - r. \tag{C.2}$$

For $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ we have $r = 18$, so $(h_{\text{ut}}^{1,1}, h_{\text{ut}}^{2,1}) = (19, 3)$ at this stage, after resolution of a single \mathbb{Z}_2 .

The integer a determines the Hodge numbers in the orbifold twisted sector. In this sector, the blow-ups and complex structure deformations of the orbifold singularities contribute

$$\begin{aligned} h_{\text{tw}}^{1,1} &= 4 + 2r - 2a = 4(k + 1), \\ h_{\text{tw}}^{2,1} &= 44 - 2r - 2a = 4g. \end{aligned} \tag{C.3}$$

Let us focus on the rightmost expressions. The factors of 4 arise since there are four fixed points on the base $\mathbb{P}^1 = T^2/\mathbb{Z}_2$. Over each fixed point there are $k + 1$ fixed curves: k rational curves and a single genus g curve. Each of the $k + 1$ curves contributes one Kähler modulus and the genus g curve contributes g complex structure deformations. It can be shown that once r is specified, k and g are not independent. They can be parametrized in terms of a single integer a , with $r - a$ even, as $k = \frac{1}{2}(r - a)$ and $g = \frac{1}{2}(22 - r - a)$.²⁷

For $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ this gives 8 fixed \mathbb{P}^1 s over each fixed point. In terms of the $T^4/(\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{P}_{(1)}^1 \times \mathbb{P}_{(2)}^1$ fiber, these can be interpreted as

$$(4 \text{ fixed points on } \mathbb{P}_{(1)}^1) \times \mathbb{P}^1 \quad \cup \quad \mathbb{P}^1 \times (4 \text{ fixed points on } \mathbb{P}_{(2)}^1).$$

From eq. (C.3), we see that the subset of Voisin-Borcea manifolds that, like $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, have $h_{\text{tw}}^{2,1} = 0$ (and hence have the same spectrum of 3-cycles on which D6 branes can be wrapped) are of the form $(r, a, \delta) = (r, 22 - r, \delta)$. Since $r \geq a$, this means that $r \geq 11$. Interestingly, the condition $r + a = 22$ corresponds to the cases that the lattice $H^2(K3, \mathbb{Z})$ cannot be primitively embedded in $(-E_8 \times E_8) \times U_{1,1}^2$ [44].²⁸ When such cases are excluded, the possible triples (r, a, δ) are symmetric about $r = 10$.

²⁷There are two exceptions to the statements in this paragraph: When $(r, a, \delta) = (10, 10, 0)$, the involution σ acts freely on $K3$ to give an Enriques surface with no fixed points. This is the FHSV case [18]. For $(r, a, \delta) = (10, 8, 0)$, we obtain the disjoint union of two elliptic curves instead of one rational curve and one elliptic curve.

²⁸There is one other exceptional case, $(r, a, \delta) = (14, 6, 0)$, for which the lattice $H^2(K3, \mathbb{Z})$ cannot be primitively embedded in $(-E_8 \times E_8) \times U_{1,1}^2$.

One additional property to note is the Euler characteristic,

$$e = 2(h^{1,1} - h^{2,1}) = 12(r - 10). \quad (\text{C.4})$$

Independent of the particular realization of the geometric twists described in section 2, we see that elements of $H^{1,1}(X)$ and $H^{2,1}(X)$ are lifted in pairs, one from each group. Thus, $|e/12| = |r - 10|$ sets a lower bound on the number of moduli that remain unlifted by the geometric twists alone. (Of course the NS and RR fluxes can lift additional moduli).

The final integer δ takes values 0 or 1. We set $\delta = 0$ if the fixed locus of σ on $K3$ is a class divisible by 2 in $H^2(K3, \mathbb{Z})$ and $\delta = 1$ otherwise. When $r \equiv 2 \pmod{4}$, both values of δ are possible and in most cases both occur, but for other r only $\delta = 1$ is possible. We refer the reader to refs. [44, 49] for a table of all possible (r, a, δ) .

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